

THE BOX GRAPH IN SUPERSTRING THEORY [†]**Eric D'Hoker** **Physics Department
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New York, N.Y. 10027, USA***ABSTRACT**

In theories of closed oriented superstrings, the one loop amplitude is given by a single diagram, with the topology of a torus. Its interpretation had remained obscure, because it was formally real, converged only for purely imaginary values of the Mandelstam variables, and had to account for the singularities of both the box graph and the one particle reducible graphs in field theories. We present in detail an analytic continuation method which resolves all these difficulties. It is based on a reduction to certain minimal amplitudes which can themselves be expressed in terms of double and single dispersion relations, with explicit spectral densities. The minimal amplitudes correspond formally to an infinite superposition of box graphs on ϕ^3 like field theories, whose divergence is responsible for the poles in the string amplitudes. This paper is a considerable simplification and generalization of our earlier proposal published in Phys. Rev. Lett. 70 (1993) p 3692.

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I. INTRODUCTION

The quantization of the force of gravity consistent with causality, Lorentz invariance, unitarity and renormalizability or finiteness is one of the outstanding problems of contemporary theoretical physics. The string picture of elementary particles in the critical dimension has provided exciting developments towards such a goal. Critical superstring theory automatically contains a massless graviton, as well as other particles, which interact consistently with general coordinate invariance. † The rules of superstring perturbation theory ensure causality, unitarity and Lorentz covariance [3,4], and there are strong indications that the scattering amplitudes are finite to all orders in perturbation theory [1,2,3,4,5].

Unlike in quantum field theory though, the rules of perturbation theory do not “manifestly” exhibit finiteness or renormalizability. In quantum field theory, simple scaling arguments and recursive combinatorics guarantee a simple physical picture of the property of renormalizability in perturbation theory. Feynman’s $i\epsilon$ prescription on the particle propagators, together with Cutkowsky and Landau cutting rules guarantee a simple picture of the property of unitarity [6]. In string theory, unitarity of the amplitudes could be established only indirectly, by showing equivalence of the Lorentz covariant formulation with a manifestly unitary, but not manifestly Lorentz invariant formulation in the light-cone gauge [3]. For the convergence issue, no such indirect understanding is available at present.

In fact, even the nature of the most basic one-loop amplitude, say in the Type II theory, for the scattering of four massless bosons, such as the graviton, antisymmetric tensor and dilaton had not, until recently [7], been completely elucidated. Following the rules of Lorentz covariant superstring perturbation theory, one finds a one-loop amplitude represented by an integral over the positions of vertex operators of incoming and outgoing states and moduli on the torus [1,2,8,9]. This representation is modular invariant, but the integral appears to be absolutely convergent only when the Mandelstam variables s , t and u are all purely imaginary [7,10,11]. For real external momenta and in fact for any values of s , t and u that are not purely imaginary, it is easy to see that the integral is real and infinite [7,12].

Both the reality and the divergence of the one-loop amplitude are physically unacceptable. The imaginary part of the (forward) one-loop amplitude is related – by the optical theorem – to the absolute value square of the tree level four point function. Reality of the one-loop amplitude for real momenta would imply the vanishing of the tree level four point amplitude, which is in contradiction with its known non-trivial expression. In fact, reality and divergence of the integral representation are manifestations of the same illness. The integral representation derived from the rules of string perturbation theory has not been properly analytically continued in its dependence on external momenta. One would arrive at the same circumstance in quantum field theory if one were to omit the $i\epsilon$ prescription in the Feynman propagators. Amplitudes constructed in standard perturbation theory rules would now be real, but cease to be convergent as every frequency integration would

† For general reviews see [1,2].

encounter a pole exactly on the real axis. In effect, the $i\epsilon$ prescription in quantum field theory instructs one on how to perform the analytic continuation.

In string theory, the equivalent of the $i\epsilon$ prescription is understood only at a formal level through the connection with the light-cone gauge formulation [3,4]. Basically, the difficulty resides in the fact that it is not so natural to exhibit all intermediate string propagators in a dual diagram, though string field theory may well be able to produce such a representation [13].

Another peculiar feature of string theory is that at each loop level, there is a single diagram, which must then account for the singularities of both the box and the one particle reducible diagrams in field theory. This feature is a manifestation of duality and points to the subtleties inherent to an $i\epsilon$ prescription in string theory.

In two recent letters [7], we have presented the crucial steps and basic arguments for the construction of the analytic continuation of these integral representations. Specifically, we considered the amplitude for the scattering of four external massless bosons, including the graviton, dilaton and anti-symmetric tensor to 1-loop in the Type II superstring. This is the simplest non-vanishing on-shell loop amplitude, and the first non-trivial quantum loop amplitude which is both finite and unitary.

In the present paper, we shall generalize the discussions and proof of [7], derive the double spectral density for all branch cuts, and present some of the proofs of the results stated in the second reference of [7]. Arguments are presented in a simplified manner, which should also allow for a reasonably straightforward generalization to the case of higher point functions to one loop and to the case of higher loops, though the latter is further complicated by the presence of intricate spin structure issues. We show that an analytic continuation exists, and we presented explicit formulas for the singularities in the complex momentum plane, in the form of poles in s , t and u and branch cuts along the positive real axis. Specifically, these topics are organized as follows.

In §II, we recall standard results about tree-level and one-loop Type II superstring amplitudes. We discuss the convergence properties of the integral representation and stress that absolute convergence is achieved only for purely imaginary values of the Mandelstam parameters. We argue that assuming Lorentz invariance and causality of the amplitudes, S-matrix elements should be analytic functions of the external momenta. * The problem of properly extending the definition of the amplitudes to physical momenta – for which the Mandelstam variables are real – thus reduces to constructing an analytic continuation in external momenta of the integral representations. The essence of this paper is in the proof that such a continuation exists and in the explicit construction of the analytically continued amplitudes. By contrast, we point out that the integral representation for the analogous amplitude in the bosonic string is nowhere convergent, and thus cannot allow for a unique analytic extension. From this point of view, the bosonic string amplitudes remain ill-defined.

In §III, we divide up the integration over moduli of the torus – including the positions of the vertex operators – into 3 inequivalent regions which transform into one another

* This hypothesis was elevated to an axiom in analytic S-matrix theory [6], but as we shall establish here, follows from string theory.

under duality [11,14]. We exhibit a formal equivalence of the amplitudes with an infinite sum over ϕ^3 quantum field theory box diagrams in which each propagator can have a different mass. This formal equivalence is invalidated though by the fact that the sum is not uniformly convergent, ** which will be intimately related to the stringy nature of the amplitudes and to duality. Next, we carefully isolate the *minimal factors* that provide an obstruction to a uniform expansion, so that the remaining factors admit a uniform Taylor series expansion. This separation is one of the crucial steps in our constructions.

In §IV, the integral of each *minimal factor* – as defined above – in turn is analytically continued. We do this by re-expressing the integral in terms of a double dispersion relation – very analogous to the Mandelstam representation of [16] – for which the spectral density and its support are explicitly known. Once such a dispersion relation is obtained and proper convergence has been established, the analytic continuation problem becomes a straightforward one, and the analytic properties of the *minimal amplitudes* may be read off directly from the dispersion representation. The analytic continuation of these *minimal factors* and the construction of the double dispersion representations and the explicit formulas for the spectral densities provides the next crucial step in obtaining the analytically continued amplitudes.

In §V, we show that all one particle intermediate state contributions – which are well-known to arise in string theory – are contained in the double dispersion representation of the minimal factors. Their appearance is shown to be directly tied into the lack of absolute convergence of the series in terms of field theory box graphs, which is resolved by retaining the *minimal factors*. Explicit formulas for residues are obtained along the way.

In §VI, we summarize the analogous analytic continuation results for Heterotic superstrings [17].

In §VII, we present some immediate applications of our results which follow from direct use of the analytically continued scattering amplitudes. These include a precise and consistent $i\epsilon$ prescription for the scattering amplitudes, a complete determination of the decay rate of massive superstrings that couple in the four-point function into 2 string states of lesser mass, and finally, an explicit discussion of the mass shift to one loop order for the lowest mass state. †

There are six appendices, organized as follows. In §A, we present and prove a number of results on analytic continuation of integrals on fixed tori, with integrations over the positions of the vertex operators only. The analytic behavior of this kind of contribution is meromorphic in the Mandelstam variables. In §B, we present a compendium of formulas related to hypergeometric functions and their integral transforms, which are crucial to the analyticity properties of the *minimal factors*. In §C, we exhibit explicit formulas for the expansion of the terms in a uniformly convergent series. In §D, we review the analytic structure of the ϕ^3 field theory box graph with arbitrary masses, and relate the support of the double spectral density to the ones we find in string amplitudes. In §E,

** A different proposal for constructing the on-shell amplitudes is given in [15], but this difficulty is not addressed there.

† Early dual model calculations of asymptotics of decay rates can be found in [18], while more recent results on decay widths and mass shifts are in [19].

we present a more direct analytic continuation method that is particularly convenient for the continuation of the forward amplitudes with $t = 0$. Finally, in §F, we give some useful analytic continuation formulas, not yet covered by the preceding appendices.

The case of amplitudes with more than four external particles is currently under investigation, and proceeds along completely analogous lines [20].

II. STRING AMPLITUDES THROUGH ANALYTIC CONTINUATION

In this section we present the integral representations of the superstring amplitudes to one-loop provided by perturbation theory. For the 4-point function, they are given by integrals over the moduli space of Riemann surfaces with four punctures and the number of handles equal to the loop order of perturbation theory. The integrand is built out of Green's functions. We discuss the convergence of these representations, and explain why analytic continuation is needed for physical values of the external momenta. We also compare the issue of causality and analyticity of the S-matrix in string theory with that of field theory.

The amplitudes $A_\ell(k_i, \epsilon_i)$ for the scattering of four external massless bosons of momenta k_i^μ and polarization tensor $\epsilon_i^{\mu\nu}(k_i)$ with $\ell = 0$ or $\ell = 1$ loops in the Type II superstring are of the form [8]

$$A_\ell(k_i, \epsilon_i) = (2\pi)^{10} \delta(k) g^4 A_\ell(s, t, u) K_{\mu_1 \mu_2 \mu_3 \mu_4} K_{\bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_3 \bar{\mu}_4} \prod_{i=1}^4 \epsilon_i^{\mu_i \bar{\mu}_i}(k_i) \quad (2.1)$$

Here g is the string coupling constant, $k = k_1 + k_2 + k_3 + k_4$ and the external states are characterized by the on-shell conditions *

$$k_i^\mu k_i^\mu = 0 \quad k_i^\mu \epsilon_i^{\mu\nu}(k_i) = k_i^\nu \epsilon_i^{\mu\nu}(k_i) = 0 \quad (2.2)$$

The remaining kinematical invariants can be expressed in terms of the Mandelstam variables (with $s + t + u = 0$)

$$\begin{aligned} s &= s_{12} = s_{34} = -(k_1 + k_2)^2 \\ t &= s_{23} = s_{14} = -(k_2 + k_3)^2 \\ u &= s_{13} = s_{24} = -(k_1 + k_3)^2 \end{aligned} \quad (2.3)$$

The kinematical factor K is a polynomial in momenta and given by

$$\begin{aligned} K_{\mu_1 \mu_2 \mu_3 \mu_4} &= -(st\eta_{13}\eta_{24} + su\eta_{14}\eta_{23} + tu\eta_{12}\eta_{34}) \\ &\quad + s(k_1^4 k_3^2 \eta_{24} + k_2^3 k_4^1 \eta_{13} + k_1^3 k_4^2 \eta_{23} + k_2^4 k_3^1 \eta_{14}) \\ &\quad + t(k_2^1 k_4^3 \eta_{13} + k_3^4 k_1^2 \eta_{24} + k_2^4 k_1^3 \eta_{34} + k_3^1 k_4^2 \eta_{12}) \\ &\quad + u(k_1^2 k_4^3 \eta_{23} + k_3^4 k_2^1 \eta_{14} + k_1^4 k_2^3 \eta_{34} + k_3^2 k_4^1 \eta_{12}) \end{aligned} \quad (2.4)$$

* Repeated Lorentz indices are to be summed over throughout, whereas repeated particle identification indices should not be summed over.

Here superscripts on the momenta label the external string states, and Lorentz indices μ_i have been abbreviated by subscripts i only. This amplitude is schematically shown in Fig. 1. The crucial factors in (2.1) are thus $A_\ell(s, t, u)$, which we describe next.

Tree Level Amplitudes

To tree-level, the amplitude is given by an integral representation derived in [8] which can be expressed in terms of Γ -functions :

$$\begin{aligned} A_0(s, t, u) &= -\frac{2}{u^2} \int d^2 z |z|^{-s-2} |1-z|^{-u} \\ &= \pi \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1+s/2)\Gamma(1+t/2)\Gamma(1+u/2)} \end{aligned} \quad (2.5)$$

Meromorphicity of the Γ function with simple poles at negative or zero integers implies the appearance of simple poles in s , t or u at positive even integers in A_0 . Physically, these poles correspond to intermediate physical massive string states and residues are given by

$$A_0(s, t, u) \underset{s \rightarrow 2n}{\sim} \frac{1}{s-2n} \frac{8\pi}{t^2} \{C_n(t)\}^2 \quad C_n(t) \equiv \frac{\Gamma(t/2+n)}{\Gamma(t/2)\Gamma(n+1)} \quad (2.6)$$

The function $\frac{2}{t}C_n(t)$ corresponds to the tree-level three-point amplitude of an intermediate state of mass $2n$, and two external massless states, and will play an important role also in our study of the one-loop amplitudes.

It is important to realize though that even this simplest situation required an analytic continuation in s, t and u . As it stands, the integral in (2.5) has singularities at $z = 0, 1, \infty$, and is absolutely convergent only for

$$\text{Re}(s) < 0, \quad \text{Re}(t) < 0 \quad \text{Re}(s) + \text{Re}(t) > -2 \quad (2.7)$$

In this region, $A_0(s, t, u)$ is holomorphic. The poles and the definition of A_0 beyond this region must be obtained through analytic continuation. The expression (2.5) in terms of Γ -functions can be viewed as a solution to this analytic continuation problem.

Without the explicit Γ -function formulas, we can still construct an analytic continuation for the integral (2.5) in the following way. By setting $x = |z|$ in (2.5), we can reduce the problem to the analytic continuation of a Mellin transform, given by the integral

$$\int_0^\delta dx x^{-1-s} f(x^2) \quad (2.8)$$

where f is a smooth function, and δ is a fixed positive cut-off. The integral (2.8) is absolutely convergent for $\text{Re}(s) < 0$ and is holomorphic in this region. If we expand $f(x^2)$ in a finite Taylor expansion with remainder $x^{2N}R_N(x^2)$, the analytic continuation of this integral to a larger region $\text{Re}(s) < 2N$ is given by

$$\int_0^\delta dx x^{-1-s} f(x^2) = - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} \frac{\delta^{2n-s}}{s-2n} + \int_0^\delta dx x^{-1+2N-s} R_N(x^2) \quad (2.9)$$

The remainder integral is absolutely convergent for $\text{Re}(s) < 2N$ and thus holomorphic in this region. The pole terms are the only singularities of $A_0(s, t, u)$ in s for $\text{Re}(s) < 2N$. Their residues are $-f^{(n)}(0)/n!$. The number N is arbitrary in this argument and so given a region, we can find all singularities in this region, plus a holomorphic remainder term. This simple example will also serve as a prototype for the more involved analytic continuations to be carried out in this paper.

One loop Type II Amplitudes

To one-loop level, the amplitude was first derived by Green and Schwarz [8,9], and admits the integral representation

$$A_1(s, t, u) = \int_F \frac{d^2\tau}{\tau_2^2} \int_{M_\tau} \prod_{i=1}^4 \frac{d^2z_i}{\tau_2} \prod_{i<j} \exp\left(\frac{1}{2}s_{ij}G(z_i, z_j)\right) \quad (2.10)$$

This integral arises from the insertion of four vertex operators at points z_i on the torus M_τ , with corners at $0, 1, \tau$ and $1 + \tau$. Here $\tau = \tau_1 + i\tau_2$ is the complex modulus of the torus, and runs through the fundamental domain

$$F = \left\{ \tau \in \mathbf{C}, \quad \tau_2 > 0, \quad |\tau_1| \leq \frac{1}{2}, \quad |\tau| \geq 1 \right\} \quad (2.11)$$

The Mandelstam variables s_{ij} were defined in (2.3) and $G(z, w)$ is the Green function for the Euclidean metric on the torus M_τ , defined by

$$-2\partial_z\partial_{\bar{z}}G(z, w) = 4\pi\delta(z - w) - \frac{4\pi}{\tau_2}$$

and expressed in terms of Jacobi ϑ -functions

$$G(z, w) = -\ln \left| \frac{\vartheta_1(z - w, \tau)}{\vartheta_1(0, \tau)} \right|^2 + \frac{2\pi}{\tau_2} \{\text{Im}(z - w)\}^2 \quad (2.12)$$

An explicit evaluation of the integral (2.10) in terms of standard special functions is not known, and thus the integral representation is our only way to define the one loop amplitude.

Qualitative Analysis of Convergence

There are two types of singularities in the integral (2.10), which are schematically depicted in Fig. 2.

- (a) When $z_i \sim z_j$, the Green function diverges : $G(z_i, z_j) \sim -\ln|z_i - z_j|^2 \rightarrow +\infty$ and produces a non-integrable singularity when $\text{Re}(s_{ij}) \geq 2$. This type of singularity occurs locally on the worldsheet when two vertex operators come close together, and – like at tree level – corresponds to a massive intermediate string state.
- (b) When z_i and z_j are well separated, but $\tau_2 \rightarrow \infty$ the Green function diverges : $G(z_i, z_j) \sim -\tau_2 \rightarrow -\infty$ and produces a non-integrable singularity when $\text{Re}(s_{ij}) < 0$.

This type of singularity occurs when the torus degenerates to a thin wire where intermediate string states go on mass-shell. It is ultimately responsible for branch cuts in the amplitude.

More subtle singularities occur when simultaneously $z_i \sim z_j$ and $\tau_2 \rightarrow \infty$, but we shall see shortly that the constraints imposed by the basic types of singularities already determine completely the range of values s_{ij} for which we have convergence. In fact convergence of the singularities (a) and (b) requires respectively that $\text{Re}(s_{ij}) < 2$ and $\text{Re}(s_{ij}) \geq 0$ for all s_{ij} . But $s + t + u = 0$, and thus we must have

$$\text{Re}(s) = \text{Re}(t) = \text{Re}(u) = 0 \quad (2.13)$$

Conversely, the integral converges for purely imaginary s, t, u . Indeed the exponentials become pure phases since $G(z, w)$ is real, so that the full amplitude A_1 is bounded by the volume of the moduli space for the torus, which is finite. Thus the condition (2.13) completely specifies the values of s_{ij} for which $A_1(s, t, u)$ is well-defined through the integral representation of (2.10).

Comparison with the Bosonic String

It is instructive to compare the above convergence analysis with that for the bosonic string. Actually, expression (2.10) closely resembles the 1-loop four tachyon amplitude in the bosonic string

$$A_1^{\text{bosonic}}(s, t, u) = \int_F \frac{d^2\tau}{\tau_2^2} Z_1 \int_{M_\tau} \prod_{i=1}^4 \frac{d^2z_i}{\tau_2} \exp\left(\frac{1}{2}s_{ij}G(z_i, z_j)\right) \quad (2.14)$$

where Z_1 is the modular invariant combination of scalar and ghost determinants with their zero mode contributions included :

$$Z_1(\tau) = \tau_2^{-12} |\eta(\tau)|^{-48} \quad (2.15)$$

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

The tachyon on-shell conditions imply $s + t + u = -8$ in the bosonic case. The analysis for singularities of type (a) above is unchanged. For singularities of type (b), we have two additional contributions. First, as $\tau_2 \rightarrow \infty$, Z_1 diverges exponentially :

$$Z_1 \sim \tau_2^{-12} e^{4\pi\tau_2} (1 + \mathcal{O}(e^{-2\pi\tau_2})) \quad (2.16)$$

and second, since now $s + t + u = -8$, at least some s_{ij} has to be negative, contributing an extra exponential divergence in τ_2 . We know of course that these exponential divergences are the result of the presence of the tachyon in the spectrum. It follows that $A_1^{\text{bosonic}}(s, t, u)$ is divergent for *all* values of s, t throughout the complex plane. There is no starting point for an analytic continuation in s, t, u , and no acceptable definition for the on-shell scattering amplitude of lowest mass particles in the bosonic string.

Lack of Proper Definition of Amplitudes near Physical Momenta

The main problem of superstring loop amplitudes addressed in this paper is now clear. The integral representation (2.10) converges and thus properly defines the loop amplitude A_1 only for unphysical values of the external momenta : when s, t, u are purely imaginary. Physical momenta are when s, t, u are real, negative for the deep inelastic or Euclidean regime and positive for the physical state pair creation regime. But around real s, t, u the integral representation is divergent and does not define the amplitude A_1 .

A second related symptom of the same problem is that for real s, t, u the entire integral representation is real, and A_1 would be real. But assuming that the optical theorem holds, the imaginary part of the amplitude equals the sum over states of the absolute value squared of the tree level amplitude, and the latter is non-zero. In general, in an interacting theory, the four point function must have an imaginary part. For this reason as well, the integral representation cannot properly define the one-loop amplitude near physical momenta.

Causality and Analytic Structure of Amplitudes

In view of the shortcomings of the integral representation, pointed out above, our task is to properly define and construct a one-loop amplitude throughout the complex s, t, u planes ($u = -s - t$), including around physical values of s, t, u .

The only possibility is to analytically continue the integral representation throughout the complex planes, starting from purely imaginary s, t, u . This is what was done at tree-level. To appreciate the physical meaning of such analytic continuations, we compare the situation with that of local quantum field theory. Here causality is equivalent to locality of observable fields and this property combined with Lorentz invariance implies analytic dependence of the Green functions of the local fields on the external momentum variables. The standard way to establish analyticity is to use Lorentz invariance and locality to derive a spectral representation for the amplitudes. The simplest case is the Källen-Lehmann representation for the off-shell two point function (represented in Fig.3), say in a scalar field theory [21] :

$$G(s) = \int_0^\infty dM^2 \frac{\rho(M^2)}{s - M^2 + i\epsilon}$$

The integration is only over $M^2 \geq 0$, as the existence of tachyonic intermediate states with $M^2 < 0$ would automatically violate causality. Provided this spectral representation is convergent, it immediately follows that the two point function is analytic in s , with singularities only on the positive real axis. These singularities may be poles resulting from one-particle intermediate states or branch cuts arising from multi-particle states. In field theory, analyticity may be shown for other amplitudes as well, and in analytic S-matrix theory, analyticity has been elevated to an axiom [6].

In string theory, a formulation of the amplitudes in terms of local field observables, commuting (or anti-commuting) at space-like separations is not at present available. It is not known whether analyticity of the amplitudes can be derived solely from the requirements of Lorentz invariance and causality (or locality of the string interactions). Thus, analyticity of the string amplitudes must be established, and our goal in this paper is to do so, starting from the integral representation.

More specifically, we must address the following important questions :

- (a) Does an analytic continuation exist ?
- (b) Is it unique ?
- (c) Is the analytic continuation physically acceptable ?

We have outlined in [7] the basic methods for the construction of an analytic continuation in the cut plane $s, t, u \in \mathbf{C} \setminus \mathbf{R}_+$ with $s + t + u = 0$ and shown that there are no gaps in the domain of holomorphicity. In this paper we present all the details of the construction as well. Uniqueness is guaranteed by the fact that two analytic continuations have to agree on purely imaginary s, t and thus must be the same.

Even if the unique analytic continuation has been constructed, there still remains the question as to whether it is consistent with all physical principles. On physical grounds, we expect the singularities of the 4-point function in the Type II superstring to consist of branch cuts along the positive real axis starting at positive even integers. In addition, there may be simple and double poles in s, t, u at positive even integers, possibly on top of branch cuts. These singularities and their field theory analogues are presented in Fig. 4. The appearance of any other type of singularity or gap would violate unitarity, causality or Lorentz invariance.

III. CONTRIBUTIONS TO THE BRANCH CUTS

We describe now the method of analytic continuation for the 4-point graviton amplitude $A_1(s, t, u)$, which we reproduce here for convenience

$$A_1(s, t, u) = \int_F \frac{d^2\tau}{\tau_2^2} \int_{M_\tau} \prod_{i=1}^4 \frac{d^2z_i}{\tau_2} \prod_{i<j} \exp\left(\frac{1}{2}s_{ij}G(z_i, z_j)\right) \quad (3.1)$$

Real Time Evolution Coordinates for the 4-Point Function

By translation invariance, we set $z_4 = 0$. The integration region in the z_i 's naturally decomposes into 6 regions according to the various orderings of $0 \leq \text{Im}(z_1), \text{Im}(z_2), \text{Im}(z_3) \leq \tau_2$. The contributions to the amplitude are the same for pairs of regions, and we obtain [7,11,14]

$$A_1(s, t, u) = 2A_1(s, t) + 2A_1(t, u) + 2A_1(u, s) \quad (3.2)$$

with $A_1(s, t)$ given by (3.1), but with the restriction

$$0 \leq \text{Im}(z_1) \leq \text{Im}(z_2) \leq \text{Im}(z_3) \leq \tau_2 \quad (3.3)$$

and $u = -s - t$. Thus it suffices to analytically continue $A_1(s, t)$, and this is the problem to which we can now restrict ourselves .

We need to express the integrand of $A_1(s, t)$ in a form where its singularities as $z_i - z_j \rightarrow 0$ and as $\tau_2 \rightarrow \infty$ can be readily extracted. Now $G(z, w)$ is given by (2.12) and $\vartheta_1(z, \tau)$ admits the following product expansion (here we set as usual $q = \exp\{2\pi i\tau\}$)

$$\vartheta_1(z, \tau) = -iq^{1/8}e^{-i\pi z} \prod_{n=0}^{\infty} (1 - q^n e^{2\pi i z})(1 - q^{n+1} e^{-2\pi i z})(1 - q^n) \quad (3.4)$$

In view of overall momentum conservation, we may ultimately ignore all z_i independent factors arising from (3.4) in the product $\prod_{i < j} \exp(-\frac{1}{2}s_{ij}G(z_i, z_j))$. Thus the Green's function $G(z, 0)$ can effectively be rewritten as

$$\exp(-G(z, 0)) = \exp\left(-\frac{2\pi}{\tau_2}(\text{Im}(z))^2 - 2\pi \text{Im}(z)\right) \left| \prod_{n=0}^{\infty} (1 - q^n e^{2\pi i z})(1 - q^{n+1} e^{-2\pi i z}) \right|^2 \quad (3.5)$$

It is convenient to introduce the following variables w_{ij}

$$w_{ij} = \begin{cases} e^{2\pi i(z_i - z_j)}, & \text{Im}(z_i - z_j) > 0 \\ qe^{2\pi i(z_i - z_j)}, & \text{Im}(z_i - z_j) < 0 \end{cases} \quad (3.6)$$

for $i \neq j$. This definition has been arranged so that $|w_{ij}| \leq 1$ in the region (3.3) and $w_{ij}w_{ji} = q$ for any $i \neq j$. The infinite products in $\prod_{i < j} \exp(-\frac{1}{2}s_{ij}G(z_i, z_j))$ resulting from the infinite product in (3.5) can now be gathered in a single expression $\mathcal{R}(w_{ij})$

$$\mathcal{R}(w_{ij}) = \prod_{i \neq j} \prod_{n=0}^{\infty} |1 - w_{ij}q^n|^{-s_{ij}} \quad (3.7)$$

To proceed further we need the real vertex coordinates (x, y) defined by

$$z = x + \tau y \quad (3.8)$$

Thus $0 \leq x, y \leq 1$, and we parametrize all tori M_τ by the same square of unit size. Set now

$$\begin{aligned} u_1 &= y_1 & \alpha_1 &= 2\pi(x_1 + u_1\tau_1) \\ u_2 &= y_2 - y_1 & \alpha_2 &= 2\pi(x_2 - x_1 + u_2\tau_1) \\ u_3 &= y_3 - y_2 & \alpha_3 &= 2\pi(x_3 - x_2 + u_3\tau_1) \\ u_4 &= 1 - y_3 & \alpha_4 &= 2\pi\tau_1 - \alpha_1 - \alpha_2 - \alpha_3 \end{aligned} \quad (3.9)$$

Evidently $0 \leq u_i \leq 1$, $i = 1, \dots, 4$ in the region of integration (3.3) for $A_1(s, t)$. The advantage of the variables u_i 's is that the exponentials on the right hand side of (3.5) combine into a remarkably simple expression when we form $\prod_{i < j} \exp(-\frac{1}{2}s_{ij}G(z_i, z_j))$

$$\prod_{i < j} \exp(-\frac{1}{2}s_{ij}G(z_i, z_j)) = |q|^{-(su_1u_3 + tu_2u_4)} \mathcal{R}(w_{ij}) \quad (3.10)$$

Since the original complex variables z_i 's, $i = 1, 2, 3$, can be replaced by any three of the u_i 's and any three of the α_i 's, we arrive at the following expression for the amplitude $A_1(s, t)$ which we take as starting point for our analytic continuation method:

$$A_1(s, t) = \int_F \frac{d^2\tau}{\tau_2^2} \int_0^{2\pi} \prod_{i=1}^4 \frac{d\alpha_i}{2\pi} \delta(2\pi\tau_1 - \sum_{i=1}^4 \alpha_i) \int_0^1 \prod_{i=1}^4 du_i |q|^{-(su_1u_3+tu_2u_4)} \mathcal{R}(w_{ij}) \quad (3.11)$$

Isolating Branch Cuts

We have seen in the discussion of the tree-level string amplitudes §II that it is not difficult to construct analytic continuations when they are just meromorphic in s and t . Thus our method for constructing an analytic continuation for $A_1(s, t)$ consists of two steps. In the first, we strip away all meromorphic terms, reducing the integrand of $A_1(s, t)$ to a much simpler expression; in the second, an explicit analytic continuation for the remaining integral is provided under the form of a double dispersion relation. This dispersion relation is the only expression exhibiting branch cuts. The stripping process has to be done with some care, since there are subtle convergence problems which reflect precisely the stringy aspects of the scattering process.

The precise statement that we need is the following. First we note that

$$q = \prod_{i=1}^4 |q|^{u_i} e^{i\alpha_i} \quad (3.12)$$

Similarly all the w_{ij} 's can be expressed in terms of products of $|q|^{u_i} e^{i\alpha_i}$

$$\begin{aligned} w_{21} &= |q|^{u_2} e^{i\alpha_2}, & w_{14} &= |q|^{u_1} e^{i\alpha_1}, & w_{32} &= |q|^{u_3} e^{i\alpha_3}, & w_{43} &= |q|^{u_4} e^{i\alpha_4} \\ w_{13} &= |q|^{u_1+u_4} e^{i\alpha_1+\alpha_4}, & w_{24} &= |q|^{u_2+u_4} e^{i\alpha_2+\alpha_4} \end{aligned} \quad (3.13)$$

and we may view $\mathcal{R}(w_{ij})$ as a function on $|q|^{u_i}, e^{i\alpha_i}, i = 1, \dots, 4$

$$\mathcal{R}(w_{ij}) = \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) \quad (3.14)$$

Of particular importance are the following four factors occurring in the infinite product expansion for $\mathcal{R}(|q|^{u_i}, \alpha_i; s, t)$

$$\prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \quad (3.15)$$

where we have introduced the notation $s_i = s$ for i even and $s_i = t$ for i odd. All other factors besides (3.15) contain $w_{ij} q^n$ which contain at least two factors $|q|^{u_i} e^{i\alpha_i}$ when

reexpressed in terms of these variables. We now define the coefficients $P_{\{n_i \nu_i\}}^{(4)}(s, t)$ as the ones occurring in the series expansion for all these remaining factors *

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{\{n_i \nu_i\}}^{(4)}(s, t) \prod_{i=1}^4 |q|^{n_i u_i} e^{i\nu_i \alpha_i} \quad (3.16)$$

The coefficients $P_{\{n_i \nu_i\}}^{(4)}(s, t)$ are polynomials in s and t , and can be generated recursively by the relations given in Appendix C.

Theorem 1. *For any positive integer N , we can write the partial amplitude as*

$$A_1(s, t) = M_N(s, t) + \sum_{n_1 + \dots + n_4 \leq 4N} \sum_{|\nu_i| \leq n_i} P_{\{n_i \nu_i\}}^{(4)}(s, t) A_{\{n_i \nu_i\}}(s, t) \quad (3.17)$$

where $M_N(s, t)$ is a meromorphic function in the region $\text{Re}(s), \text{Re}(t) < N$, and the amplitudes $A_{\{n_i \nu_i\}}(s, t)$ are defined by

$$\begin{aligned} A_{\{n_i \nu_i\}}(s, t) = & \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^{2\pi} \int_0^1 \prod_{i=1}^4 \frac{d\alpha_i du_i}{2\pi} \delta(1 - \sum_{i=1}^4 u_i) |q|^{-su_1 u_3 - tu_2 u_4} \\ & \times \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} |q|^{n_i u_i} e^{i\nu_i \alpha_i} \end{aligned} \quad (3.18)$$

Thus Theorem 1 reduces the analytic continuation of the amplitude $A_1(s, t)$ to the analytic continuation of the much simpler amplitudes $A_{\{n_i \nu_i\}}(s, t)$, where there are no longer infinite products, and where all angle α_i -dependence is decoupled.

Box Diagrams in Field Theory and String Theory

Before giving the proof of Theorem 1, we pause to discuss carefully the convergence issues behind the *partial* expansion (3.16), issues which prevent us from using the *full* expansion of all the factors in $\mathcal{R}(|q|^{u_i}, \alpha_i; s, t)$. The situation can be illustrated clearly by a comparison with the box diagrams in a ϕ^3 -like field theory. Suppose we had expanded all factors in $\mathcal{R}(|q|^{u_i}, \alpha_i; s, t)$ into a series in $|q|^{u_i} e^{i\alpha_i}$

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{\{n_i \nu_i\}}^{(0)}(s, t) \prod_{i=1}^4 |q|^{n_i u_i} e^{i\nu_i \alpha_i} \quad (3.19)$$

* The upper index (4) is a reminder of how many factors in \mathcal{R} are not expanded in the Taylor series. Later on we shall also use coefficients with a different number of factors retained.

The coefficients $P_{\{n_i \nu_i\}}^{(0)}$ are still polynomials in s and t . Formally, this expansion leads to an expansion for $A_1(s, t)$ into terms of the form

$$A_{\{n_i \nu_i\}}^{(0)}(s, t) = \int_F \frac{d^2 \tau}{\tau_2^2} \int_0^{2\pi} \prod_{i=1}^4 \frac{d\alpha_i}{2\pi} \delta(2\pi\tau_1 - \sum_{i=1}^4 \alpha_i) \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) \times |q|^{-su_1 u_3 - tu_2 u_4} \prod_{i=1}^4 |q|^{n_i u_i} e^{i\nu_i \alpha_i} \quad (3.20)$$

We shall see later that we may truncate the τ domain of integration F in this integral to the simpler domain $\{\tau_2 \geq 1, |\tau_1| \leq 1/2\}$ without affecting the branch cuts nor the poles lying on top of them. The variables τ_1 and α_i , $i = 1, \dots, 4$ can then be all integrated out, leaving an integral of the form

$$\int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) e^{2\pi\tau_2(su_1 u_3 + tu_2 u_4 - \sum_{i=1}^4 n_i u_i)} \quad (3.21)$$

where the integers n_i are positive and even.

Consider now a ϕ^3 -like box diagram in d dimensional space-time quantum field theory with masses m_i^2 for each of the propagators, and massless external on-shell states, as depicted in Fig. 5. Couplings are identical for all m_i , and are non-derivative ϕ^3 . The box diagram (Euclidian) Feynman integral can be performed as usual after introducing Feynman parameters u_i and exponentiating the denominator (see Appendix D). We obtain

$$\int d^d k \prod_{i=1}^4 \frac{1}{(k + p_i)^2 + m_i^2} = \int_0^\infty \frac{d\tau}{\tau^{\frac{d}{2}-3}} \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) e^{2\pi\tau(su_1 u_3 + tu_2 u_4 - \sum_{i=1}^4 u_i m_i^2)} \quad (3.22)$$

In space-time dimension $d = 10$ and with the masses m_i^2 positive even integers (as dictated by the superstring spectrum), the ϕ^3 -like box diagram is essentially the same as the partial superstring amplitude of (3.21). A well-understood difference is that the $d\tau_2$ integral in the superstring amplitude is truncated away from 0. This is a remnant of duality [9]. Duality implies modular invariance, which restricts the domain of integration in $A_1(s, t)$ to a fundamental domain for $SL(2, \mathbf{Z})$, and insures ultra-violet finiteness. Of interest to us is rather the issue of how the poles on top of branch cuts expected in string theory (and absent in an infinite superposition of ϕ^3 like theories) can emerge. The fact that the summands are of the same form and have no poles shows that there must be difficulties in resumming the series resulting from (3.19). These convergence issues are important manifestations of the difference between string theory and mere infinite superpositions of field theories.

Singularities of the Green's function

We turn now to a precise analysis of the problem of convergence, leading ultimately to the proof of Theorem 1. First the product expansion for the Green's function can be expressed in terms of the real coordinates (x, y) as

$$\exp(-G(z, 0)) = \frac{1}{4\pi^2} |q|^{y^2-y} |1 - q^y e^{2\pi i x}|^2 |1 - q^{1-y} e^{-2\pi i x}|^2 R(z) \quad (3.23)$$

with

$$R(z) = \prod_{n=1}^{\infty} |1 - q^{n+y} e^{2\pi i x}|^2 |1 - q^{n+1-y} e^{-2\pi i x}|^2 |1 - q^n|^{-4} \quad (3.24)$$

Since $|q| < e^{-2\pi\sqrt{3}}$ in the fundamental domain F , the function $R(z)$ is uniformly bounded from above and below in all variables. Thus the singularities of $G(z, w)$ as $z \rightarrow w$ and/or $\tau \rightarrow \infty$ are described precisely by the first three factors in (3.23). The expansion (3.23) leads to a further splitting in the factorization (3.10) for the integrand of $A_1(s, t)$

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \mathcal{I}(|q|^{u_i}, \alpha_i; s, t) \mathcal{R}^*(|q|^{u_i}, \alpha_i; s, t) \quad (3.25)$$

with

$$\begin{aligned} \mathcal{I}(|q|^{u_i}, \alpha_i; s, t) &= \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} |1 - \prod_{j \neq i} e^{i\alpha_j} |q|^{u_j}|^{-s_i} \\ &\quad \times \prod_{i=2k} \prod_{j=2m+1} |1 - e^{i(\alpha_i + \alpha_j)} |q|^{u_i + u_j}|^{s_i + s_j} \\ \mathcal{R}^* &= \left(\frac{R(z_2 - z_1)R(z_3)}{R(z_3 - z_1)R(z_2)} \right)^{-s} \left(\frac{R(z_3 - z_2)R(z_1)}{R(z_3 - z_1)R(z_2)} \right)^{-t} \end{aligned} \quad (3.26)$$

As in the case of (3.24) for the Green's function, \mathcal{R}^* is uniformly bounded from above and below in all variables, and the 12 factors in \mathcal{I} are the only ones incorporating singularities in \mathcal{R} . The expansion (3.25) shows immediately that the integrations over the vertex operator locations in $A_1(s, t)$ converges only in the infinite strip

$$\text{Re}(s), \text{Re}(t) \leq 0, \text{Re}(s+t) > -2 \quad (3.27)$$

Conversely, in this region, the last factor $|q|^{-(su_1u_3+tu_2u_4)}$ remains bounded as $\tau_2 \rightarrow \infty$, so that the $d^2\tau$ integral is finite as well. Thus we have established that (3.27) is the domain of convergence of $A_1(s, t)$. Note, however, that all three expressions on the right hand side of (3.1) - and hence the full amplitude $A_1(s, t, u)$ - are simultaneously convergent only when $\text{Re}(s) = \text{Re}(t) = \text{Re}(u) = 0$ when the constraint $s+t+u=0$ is enforced. This confirms what we foresaw in §II from a mere qualitative analysis.

Proof of Theorem 1

The factorization (3.25)-(3.26) shows that the full factor \mathcal{R} cannot be expanded as a uniformly convergent series in the variables $|q|^{u_i} e^{i\alpha_i}$, as we had anticipated on physical grounds in our earlier comparison of string theory with field theory. The only factor which can be expanded uniformly is the factor \mathcal{R}^* , which leaves us with the seemingly intractable 12 factors of \mathcal{I} . The key observation behind Theorem 1 is that actually the eight factors in \mathcal{I} involving *composites* of $|q|^{u_i} e^{i\alpha_i}$ can also be expanded, at the harmless expense of discarding purely meromorphic terms which do not affect the behavior of the amplitude near the branch cuts.

To see this, we begin by observing that for fixed τ , the integrals $du_i d\alpha_i$ over the location of the vertex operators always produce globally meromorphic functions of s and t , with at most double poles in s , t , and $u = -(s+t)$ at integers ≥ 2 . Indeed, for fixed τ , the factor $|q|^{-(su_1 u_3 + tu_2 u_4)} \mathcal{R}^*$ is a smooth, bounded function of the u_i, α_i , and only \mathcal{I} plays a role in determining convergence of the $du_i d\alpha_i$ integrals. Setting $w_i = |q|^{u_i} e^{i\alpha_i}$, these integrals can be viewed as integrals over a subdomain of $|w_i| \leq 1$. In view of the constraint $|q|^{u_1 + u_2 + u_3 + u_4} = |q| < e^{-\pi\sqrt{3}}$, at most 6 of the 12 factors in \mathcal{I} can approach 0. Thus the $du_i d\alpha_i$ integrals reduce to integrals in the complex variables w_i 's of the form

(i)

$$\int_{\frac{1}{2} \leq |w_1| \leq 1} d^2 w_1 |1 - w_1|^{-s} E(w_1)$$

(ii)

$$\int_{\frac{1}{2} \leq |w_2| \leq |w_1| \leq 1} d^2 w_1 d^2 w_2 |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_1 w_2|^{s+t} E(w_1, w_2)$$

(iii)

$$\int_{\substack{\frac{1}{2} \leq |w_i| \leq 1 \\ |w_{i+1}| \geq |w_i|}} \prod_{i=1}^3 d^2 w_i |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_3|^{-t} |1 - w_1 w_2|^{s+t} \\ \times |1 - w_2 w_3|^{s+t} |1 - w_1 w_2 w_3|^{-s} E(w_1, w_2, w_3)$$

Here $E(w_i)$ are smooth bounded functions which incorporate $|q|^{-su_1 u_2 - tu_2 u_4}$, \mathcal{R}^* as well as well-behaved factors in \mathcal{I} . Our claim follows from the analytic continuations of integrals of the form (i)-(iii) derived in Appendix A.

The meromorphicity of the contribution of each fixed τ implies the meromorphicity of the contribution of any bounded region in τ . In particular the region $\{|\tau_1| \leq 1/2, |\tau| \geq 1, \tau_2 \leq 1\}$ contributes only a meromorphic term which can be absorbed in the expression $M_N(s, t)$ in Theorem 1. We may restrict then the τ domain of integration to the semi-infinite strip

$$\{|\tau_1| \leq 1/2, \tau_2 \geq 1\} \quad (3.28)$$

in which the variable $\alpha_4 = 2\pi\tau_1 - \alpha_1 - \alpha_2 - \alpha_3$ can be treated as an angle on the same footing as the other three angle variables $\alpha_1, \alpha_2, \alpha_3$. The contribution of (3.28) is then invariant under each interchange $u_1 \leftrightarrow u_2$ and $u_3 \leftrightarrow u_4$. The region of integration can thus be restricted to $u_1 \leq u_2$ and $u_3 \leq u_4$ upon including an overall factor of 4. We need to consider separately the contributions of the following two regions

$$\begin{aligned} \text{(I)} &= \{(u_1 + u_2)\tau_2 < 1\} \\ \text{(II)} &= \{(u_1 + u_2)\tau_2 \geq 1\} \end{aligned} \quad (3.29)$$

In Region I, the expression

$$|q|^{-(su_1 u_2 + tu_3 u_4)}$$

remains bounded. This implies that the contributions of each τ in this region can be summed up in a convergent integral and produce no new singularities. The careful mathematical arguments are given in Appendix A. The net outcome is that Region I just produces a meromorphic function of s and t which can be absorbed into the expression $M_N(s, t)$ allowed in Theorem 1.

In Region (II), we observe that $\tau_2(u_i + u_j) \leq 1$ whenever one subindex i or j is even and the other one is odd. Under these conditions $|q|^{u_i+u_j} \leq e^{-2\pi} < 1$ and, except for the first four factors

$$\prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \quad (3.30)$$

in the infinite product expression for \mathcal{R} , all the other factors (i.e. \mathcal{R}^* as well as 8 of the factors in \mathcal{I}) are bounded away from 0. The expansion (3.16), which is just the expansion of these non-vanishing factors in a series in $|q|^{u_i}$ and $e^{i\alpha_i}$, is then uniformly convergent. We assert now that for each N it suffices to consider a finite number of terms in this expansion in order to obtain an analytic continuation in s and t to the half-space $\text{Re}(s) < N$, $\text{Re}(t) < N$. Since N is arbitrary, we actually obtain in this way an analytic continuation to the whole s and t planes. Fix then N . The uniform convergence of the series (3.16) for \mathcal{R} allows us to rewrite it under the form of a *limited* Taylor expansion

$$\begin{aligned} \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = & \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \left(\sum_{n_1+\dots+n_4 \leq 4N} \sum_{|\nu_i| \leq n_i} P_{n_i \nu_i}(s, t) \prod_{i=1}^4 |q|^{n_i u_i} e^{i \nu_i \alpha_i} \right. \\ & \left. + \sum |q|^{n_1 u_1 + \dots + n_4 u_4} E_{n_i}(|q|^{u_i}, \alpha_i; s, t) \right) \end{aligned} \quad (3.31)$$

where the second sum on the right hand side is over indices n_i satisfying

$$n_1 + n_3 = n_2 + n_4 = N$$

and E_{n_i} are suitable smooth functions. To see (3.31) we begin by observing that any term $w_{ij} q^n$ in the factorization (3.7) for \mathcal{R} satisfies

$$|(n_1 + n_3) - (n_2 + n_4)| \leq 1$$

when written under the form $\prod_{i=1}^4 |q|^{n_i u_i} e^{i \nu_i \alpha_i}$. Consider next any product $\prod_{\ell=1}^L w_{i_\ell j_\ell} q^{n_\ell}$ where each individual factor corresponds to exponents $\{n_i^\ell, \nu_i^\ell\}$. The exponents $\{n_i \nu_i\}$ corresponding to the product will satisfy

$$|(n_1 + n_3) - (n_2 + n_4)| \leq L \quad (3.32)$$

Now except for the first four factors (3.30) which are not expanded, all other factors involve $w_{i_\ell j_\ell} q^{n_\ell}$ with

$$\min(n_1^\ell + n_3^\ell, n_2^\ell + n_4^\ell) \geq 1$$

This implies $L \leq \min(n_1 + n_3, n_2 + n_4)$, which combined with (3.32), implies in turn

$$\frac{1}{2}(n_1 + n_3) \leq n_2 + n_4 \leq 2(n_1 + n_3) \quad (3.33)$$

Returning now to the Taylor expansion of \mathcal{R} , we note that any term not included in the first sum on the right hand side of (3.31) must satisfy either $n_1 + n_3 > 2N$ or $n_2 + n_4 > 2N$. In either case, the other pair of indices must add up to more than N , in view of (3.33). Thus such terms can be absorbed into the second sum on the right hand side of (3.31).

Evidently the above truncation respects the $u_1 \leftrightarrow u_3, u_2 \leftrightarrow u_4$ symmetry. We observe now that the last term gives rise to only a meromorphic function in the given half-planes $\text{Re}(s), \text{Re}(t) < N$. Indeed

$$|q|^{-su_1u_3+n_1u_1+n_3u_3}|q|^{-tu_2u_4+n_2u_2+n_4u_4} \quad (3.34)$$

is bounded then as $\tau_2 \rightarrow \infty$. As in the case of Region I, the meromorphic contributions corresponding to these terms for each τ again add up to just a meromorphic function. Thus it suffices to treat the terms in the first sum in (3.31). These are precisely the terms stated in Theorem 1, except that the integration region has been truncated to Region II. Since Region I with the simplified integrand still produces only a meromorphic function, we may reattach it. Furthermore, if we proceed in groups of 4 terms respecting the $1 \leftrightarrow 3, 2 \leftrightarrow 4$ symmetry, we can rewrite the integral over $u_1 \leq u_3, u_2 \leq u_4$ in terms of the original full region $0 \leq u_i \leq 1, i = 1, \dots, 4$. This establishes Theorem 1.

IV. DOUBLE DISPERSION RELATIONS

With Theorem 1, the problem of analytically continuing the amplitude $A_1(s, t)$ reduces to the problem of analytically continuing $A_{\{n_i\nu_i\}}$ for each fixed n_i, ν_i . In this section, we carry this out by showing that $A_{\{n_i\nu_i\}}$ can be expressed as a double dispersion relation. Such a relation gives immediately an analytic continuation to the whole plane in s and t cut along the positive real axis. The corresponding (double) spectral density has a natural interpretation as a density of intermediate states, and we exhibit it explicitly. In particular its support can be parametrized by conics.

The minimal amplitudes $A_{\{n_i\nu_i\}}$ in terms of hypergeometric functions

The first step in our derivation of the double dispersion relations is to express the amplitudes $A_{\{n_i\nu_i\}}$ in terms of Laplace transforms of Gauss' hypergeometric functions $F(a, b; c; x)$. Recall that $A_{\{n_i\nu_i\}}$ is given by (3.18). Thus it can be rewritten as

$$A_{\{n_i\nu_i\}}(s, t) = \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 du_i \delta(1 - \sum_{i=1}^4 u_i) |q|^{-su_1u_3-tu_2u_4+\sum_{i=1}^4 u_i(n_i+|\nu_i|)} \\ \times \prod_{i=1}^4 C_{|\nu_i|}(s_i) F\left(\frac{s_i}{2}, \frac{s_i}{2} + |\nu_i|; |\nu_i| + 1; |q|^{2u_i}\right) \quad (4.1)$$

since each $d\alpha$ integral produces a hypergeometric function

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha\nu} |1 - xe^{i\alpha}|^{-s} = C_{|\nu|}(s) x^{|\nu|} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) \quad (4.2)$$

If we introduce the (inverse) Laplace transform $\varphi_{n\nu}(s; \beta)$ of $F(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2)$ by

$$\begin{aligned} C_{|\nu|}(s) x^{n+|\nu|} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) &= \int_0^\infty d\beta x^\beta \varphi_{n\nu}(s; \beta) \\ \varphi_{n\nu}(s; \beta) &= 0 \text{ for } \beta < 0 \end{aligned} \quad (4.3)$$

we immediately obtain the desired formula.

Lemma 1. *The amplitude $A_{\{n_i\nu_i\}}$ can be written as*

$$\begin{aligned} A_{\{n_i\nu_i\}}(s, t) &= \int_0^\infty \prod_{i=1}^4 d\beta_i \Psi_{\{n_i\nu_i\}}(s, t; \beta_i) \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 \prod_{i=1}^4 du_i \delta\left(1 - \sum_i u_i\right) \\ &\quad \times \exp\left\{-2\pi\tau_2 \sum_i u_i \beta_i + 2\pi\tau_2 (su_1 u_3 + tu_2 u_4)\right\} \end{aligned} \quad (4.4)$$

with

$$\Psi_{n_i\nu_i}(s, t; \beta_i) = \prod_{i=1}^4 \varphi_{n_i\nu_i}(s_i; \beta_i) \quad (4.5)$$

In particular the support of $\Psi_{n_i\nu_i}$ is contained in $\{\beta_i \geq 0, i = 1, \dots, 4\}$.

The analyticity properties of $\varphi_{n\nu}(s; \beta)$ as well as those of the closely related Mellin transform $f_{n\nu}(s, \alpha)$ of $F(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2)$ are given in Appendix B. For the moment, we note only that $\varphi_{n\nu}(s; \beta)$ is an infinite superposition of Dirac point masses or delta functions

$$\varphi_{n\nu}(s; \beta) = \sum_{k=0}^\infty C_k(s) C_{k+|\nu|}(s) \delta(2k + n + |\nu| - \beta) \quad (4.6)$$

In particular, its integrals in β over any finite interval are entire functions of s , although they can have poles if the interval is infinitely extended, as shown in Appendix B.

Derivation of the Double Dispersion Relation

To recast $A_{\{n_i\nu_i\}}$ in the form of a double dispersion relation, we start from the formula (4.4) provided by Lemma 1. The variables u_3 and u_4 are changed to

$$\sigma_3 = 2\pi\tau_2 u_3 \quad \sigma_4 = 2\pi\tau_2 u_4$$

Without loss of generality, we may extend the integration region for u_3 and u_4 , and hence for σ_3 and σ_4 to the half-line $[0, \infty]$, in view of the presence of the δ -function on the u 's

in (4.4). Having done so, we may view the δ -function now as a δ -function for the modulus τ_2 , and we can easily integrate out τ_2 . The result is

$$\begin{aligned}
A_{\{n_i \nu_i\}}(s, t) = & 2\pi \int_0^\infty \prod_{i=1}^4 d\beta_i \Psi_{\{n_i \nu_i\}}(s, t; \beta_i) \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^\infty d\sigma_3 \int_0^\infty d\sigma_4 \frac{(1-u_1-u_2)^2}{(\sigma_3 + \sigma_4)^3} \\
& \times \theta(\sigma_3 + \sigma_4 - 2\pi(1-u_1-u_2)) \\
& \times \exp \left\{ (su_1 - \beta_3)\sigma_3 + (tu_2 - \beta_4)\sigma_4 \right. \\
& \left. - \frac{\sigma_3 + \sigma_4}{1-u_1-u_2} (u_1\beta_1 + u_2\beta_2) \right\} \quad (4.7)
\end{aligned}$$

We make a further change of variables from σ_3, σ_4 to $\mu \in [0, \infty]$ and $\alpha \in [0, 1]$ by

$$\begin{aligned}
\sigma_3 &= 2\pi(1-u_1-u_2)\mu\alpha \\
\sigma_4 &= 2\pi(1-u_1-u_2)\mu(1-\alpha)
\end{aligned}$$

and arrive at

$$\begin{aligned}
A_{\{n_i \nu_i\}}(s, t) = & \int_0^\infty \prod_{i=1}^4 d\beta_i \Psi_{\{n_i \nu_i\}}(s, t; \beta_i) \int_0^1 du_1 \int_0^{1-u_1} du_2 (1-u_1-u_2) \\
& \times \int_0^1 d\alpha \int_1^\infty \frac{d\mu}{\mu^2} e^{-\mu\kappa} \quad (4.8)
\end{aligned}$$

Here we have defined the function κ , linear in s and t by

$$\begin{aligned}
\kappa = & 2\pi \left\{ u_1\beta_1 + u_2\beta_2 + (1-u_1-u_2)(\alpha\beta_3 + (1-\alpha)\beta_4) \right. \\
& \left. - s(1-u_1-u_2)u_1\alpha - t(1-u_1-u_2)u_2(1-\alpha) \right\} \quad (4.9)
\end{aligned}$$

Now we come to the actual analytic continuation. We rewrite the integral over μ in terms of the following expression

$$\int_1^\infty \frac{d\mu}{\mu^2} e^{-\mu\kappa} = \frac{1}{2} \int_0^\infty d\rho \left[\frac{\rho^2}{(\rho + \kappa)^2} - 1 + \frac{2\kappa}{\rho + 1} \right] + E(\kappa) \quad (4.9)$$

where $E(\kappa)$ is the entire function of κ given by

$$E(\kappa) = - \int_0^1 \frac{d\mu}{\mu^2} (e^{-\mu\kappa} - 1 + \kappa\mu) + (c_2 - \frac{3}{2})\kappa + 1 \quad (4.10)$$

Here c_2 is a constant. The following Lemma shows that $E(\kappa)$ does not affect the branch cuts:

Lemma 2. *The contribution of $E(\kappa)$ to the amplitude $A_{\{n_i\nu_i\}}$*

$$M_{\{n_i\nu_i\}}(s, t) = \prod_{i=1}^4 \int_0^\infty d\beta_i \Psi_{\{n_i\nu_i\}}(s, t; \beta_i) \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^1 d\alpha (1 - u_1 - u_2) E(\kappa) \quad (4.11)$$

is a globally meromorphic function of both s and t . Furthermore, its poles are at integers on the real axes in s and t , with real coefficients.

Indeed, all the expressions entering (4.11) are actually moments of $\varphi_{n\nu}$ and hence are all given by special values of the hypergeometric function, its derivatives, and its integrals at $z = 1$. Such values are meromorphic functions of s . The details are in Appendix B.

Henceforth we can ignore this part of $A_{\{n_i\nu_i\}}$, as it does not enter the double dispersion representation. We turn now to the first expression on the right hand side of (4.9). The argument is based on a few successive changes of variables. For greater clarity we work them out only on the main contribution, which is formally that of the term $\rho^2(\rho + \kappa)^{-2}$

$$\int_0^\infty \prod_{i=1}^4 d\beta_i \Psi_{\{n_i\nu_i\}}(s, t; \beta_i) \int_0^{1-u_1} du_1 (1 - u_1 - u_2) \int_0^1 d\alpha \int_0^\infty d\rho \frac{\rho^2}{(\rho + \kappa)^2} \quad (4.12)$$

Since κ is a linear function of α , the α -integral may be carried out explicitly, and (4.12) becomes

$$\begin{aligned} & \int_0^\infty \prod_{i=1}^4 d\beta_i \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 \Psi_{\{n_i\nu_i\}}(s, t; \beta_i) \\ & \times \int_0^\infty dx \frac{x^2}{(x + x_0 + \beta_3 - su_1)(x + x_0 + \beta_4 - tu_2)} \end{aligned} \quad (4.13)$$

where we made the change of variables $\rho = 2\pi(1 - u_1 - u_2)x$ and set

$$x_0 \equiv \frac{u_1\beta_1 + u_2\beta_2}{1 - u_1 - u_2} \quad (4.14)$$

Upon performing a last change of variables from β_3 and β_4 to σ and τ

$$\beta_3 = -x - x_0 + u_1\sigma \quad \beta_4 = -x - x_0 + u_2\tau \quad (4.15)$$

the main term (4.12) takes the form of a double dispersion relation

$$\int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho(s, t; \sigma, \tau)}{(s - \sigma)(t - \tau)} \quad (4.16)$$

The double spectral density $\rho(s, t; \sigma, \tau)$ in (4.16) for each minimal amplitude is given by

$$\begin{aligned} \rho_{\{n_i\nu_i\}}(s, t; \sigma, \tau) &= \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 \int_{x_0}^\infty dx (x - x_0)^2 \\ &\times \Psi_{\{n_i\nu_i\}}(s, t; \beta_1, \beta_2, -x + u_1\sigma, -x + u_2\tau) \end{aligned} \quad (4.17)$$

For given s and t , and fixed σ and τ , all the integrals in the definition of $\rho(s, t; \sigma, \tau)$ run over finite ranges. To see this, we note that from the structure of $\Psi_{\{n_i \nu_i\}}$, its first four arguments must be positive, which requires that $0 \leq x \leq \sigma, \tau$ and in view of $x_o \leq x$, also $0 \leq \beta_1 \leq \sigma$ and $0 \leq \beta_2 \leq \tau$. As a result, $\rho(s, t; \sigma, \tau)$ is a completely finite function of s, t and σ and τ . In fact for fixed σ and fixed τ it is polynomial in s and t , and the degree of this polynomial increases with σ and τ .

Next, the same calculation leads to the following formula for the contribution of the remaining term $1 - 2\kappa(\rho + 1)^{-1}$ in (4.9)

$$\int_0^\infty \int_0^\infty d\sigma d\tau \Lambda_{\{n_i \nu_i\}}(s, t; \sigma, \tau)$$

with $\Lambda_{\{n_i \nu_i\}}(s, t; \sigma, \tau)$ given by

$$\begin{aligned} \Lambda_{\{n_i \nu_i\}}(s, t; \sigma, \tau) = & \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty dx \Psi(s, t; \beta_1, \beta_2, \sigma u_1 - x_0 - x, \tau u_2 - x_0 - x) \\ & \times \left(1 - 2\pi(1 - u_1 - u_2) \frac{u_1(\sigma - s) + u_2(\tau - t) - 2x}{2\pi(1 - u_1 - u_2) + 1} \right) \end{aligned} \quad (4.18)$$

In the same way as the term $1 - 2\kappa(\rho + 1)^{-1}$ guaranteed the convergence of the $d\rho$ integral in (4.9), the role of $\Lambda_{\{n_i \nu_i\}}$ in (4.18) is as a subtraction term guaranteeing the convergence of the $d\sigma d\tau$ integrals in the final expression (4.16)+(4.18) for $A_{\{n_i \nu_i\}}$. In practice, however, it produces only meromorphic functions of s and t . Thus the behavior of $A_{\{n_i \nu_i\}}$ at the branch cuts is completely described by the double dispersion relation (4.16), and for all practical purposes, we may safely drop $\Lambda_{\{n_i \nu_i\}}$ from our considerations.

We have thus established all the statements except the last one in the following theorem:

Theorem 2. *The minimal amplitudes $A_{\{n_i \nu_i\}}(s, t)$ can be expressed as*

$$A_{\{n_i \nu_i\}}(s, t) = M_{\{n_i \nu_i\}}(s, t) + \int_0^\infty \int_0^\infty d\tau d\sigma \left(\frac{\rho_{\{n_i \nu_i\}}(s, t; \sigma, \tau)}{(s - \sigma)(t - \tau)} - \Lambda_{\{n_i \nu_i\}} \right) \quad (4.19)$$

where the density $\rho_{\{n_i \nu_i\}}$ and the meromorphic subtraction term $\Lambda_{\{n_i \nu_i\}}$ are given by (4.17) and (4.18). The term $M_{\{n_i \nu_i\}}$ is another globally meromorphic function of s and t , whose properties are described in Lemma 2. The integral on the right hand side in (4.19) defines a holomorphic function of s, t in the cut plane $s, t \in \mathbf{C} \setminus \mathbf{R}_+$. More precisely, its domain of holomorphy is given by

$$s \in \mathbf{C} \setminus [(m_1 + m_3)^2, +\infty), \quad t \in \mathbf{C} \setminus [(m_2 + m_4)^2, \infty) \quad (4.20)$$

where we have set

$$m_i^2 = n_i + |\nu_i|, \quad i = 1, \dots, 4 \quad (4.21)$$

The last statement will be established in the next section. The factorized expression (4.5) for $\Psi_{\{n_i \nu_i\}}$ in terms of four $\varphi_{n\nu}$ -functions shows that the double spectral density can also be expressed as

$$\begin{aligned} \rho_{\{n_i \nu_i\}}(s, t; \sigma, \tau) = & \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \varphi_{n_1 \nu_1}(t; \beta_1) \varphi_{n_2 \nu_2}(s; \beta_2) \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 \\ & \times \int_{x_0}^\infty dx (x - x_0)^2 \varphi_{n_3 \nu_3}(t; u_1 \sigma - x) \varphi_{n_4 \nu_4}(s; u_2 \tau - x) \end{aligned} \quad (4.22)$$

This generalizes the result presented in [7] for the case $n_i = 0, \nu_i = 0$.

Location of cuts.

The expression (4.20) for the amplitude $A_{\{n_i \nu_i\}}$ shows that the branch cuts in both s and t lie within the positive real axis. Where they begin depends however on n_i, ν_i . To address this issue, we need a closer inspection of the support of the density $\rho_{\{n_i \nu_i\}}(s, t; \sigma, \tau)$. Since the Laplace transform $\varphi_{n\nu}(s; \beta)$ of the hypergeometric function F can be expanded into a series of Dirac point masses (see (4.6)), we may also expand $\rho_{\{n_i \nu_i\}}$ into a series of “monomial” densities

$$\rho_{\{n_i \nu_i\}}(s, t; \sigma, \tau) = \sum_{k_i=0}^{\infty} \prod_{i=1}^4 C_{k_i}(s_i) C_{k_i+|\nu_i|}(s_i) \rho^{\{2k_i+n_i+|\nu_i|\}}(s, t; \sigma, \tau) \quad (4.23)$$

where the monomial density $\rho^{M_i^2}$ is defined by the same integral (4.19), with however the following choice for $\Psi(\beta_i)$

$$\Psi_{\{M_i^2\}} = \prod_{i=1}^4 \delta(\beta_i - M_i^2) \quad (4.24)$$

Thus it suffices to determine the support of each density $\rho^{\{M_i^2\}}$, for each $M_i^2 = 2k_i + n_i + |\nu_i|$. The $d\beta_i$ integrals can now be carried out, and we find

$$\begin{aligned} \rho^{\{M_i^2\}}(s, t; \sigma, \tau) = & \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 \int_{x_0}^\infty dx (x - x_0)^2 \\ & \times \delta(x + M_3^2 - \sigma u_1) \delta(x + M_4^2 - \tau u_2) \end{aligned} \quad (4.25)$$

with

$$x_0 = \frac{u_1 M_1^2 + u_2 M_2^2}{1 - u_1 - u_2} \quad (4.26)$$

These expressions coincide exactly with those for the spectral density of the box diagram for ϕ^3 field theory (c.f. (D.5) in Appendix D). Thus the parametrization (D.8) derived in Appendix D for the ϕ^3 spectral density holds in the string case as well. In particular, the support of $\rho^{\{M_i^2\}}$ as a function of σ and τ is contained in the region $A \geq B \geq 0$, where A and B are given by (D.7), with m_i^2 replaced by M_i^2 .

Let $A_{\{n_i\nu_i\}}^{\{M_i^2\}}$ denote the contribution of the monomial density $\rho^{\{M_i^2\}}$ to the amplitude $A_{\{n_i\nu_i\}}$. We begin by identifying the domain of holomorphy of $A_{\{n_i\nu_i\}}^{\{M_i^2\}}$. It is given by

$$\{s \notin \pi_\sigma(\text{support } \rho^{\{M_i^2\}})\} \cup \{t \notin \pi_\tau(\text{support } \rho^{\{M_i^2\}})\} \quad (4.27)$$

where π_σ and π_τ denote respectively the projections of (σ, τ) on the first and the second variable. Now the support of $\rho^{\{M_i^2\}}$ is determined by the condition $A \geq B \geq 0$, which can be rewritten as

$$(M_1^2 + M_3^2)\frac{1}{\sigma} + (M_2^2 + M_4^2)\frac{1}{\tau} + 2\left(\frac{1}{\sigma} + \frac{1}{\tau}\right)(M_1^2 M_3^2 \frac{1}{\sigma} + M_2^2 M_4^2 \frac{1}{\tau})^{1/2} \leq 1 \quad (4.28)$$

The left hand side defines a function f of $(\frac{1}{\sigma}, \frac{1}{\tau})$ which is increasing, in the sense that

$$f(\alpha, \beta) < f(\alpha', \beta')$$

for all $0 \leq \alpha \leq \alpha'$, $0 \leq \beta \leq \beta'$, $\alpha + \beta < \alpha' + \beta'$. Thus its contour must have the form depicted in Fig. 6. In particular for any level set $f(\alpha, \beta) = \text{constant}$, the point with the greatest α value is the point on the α axis. In the case at hand, this point is given by

$$\alpha_{M_i^2} = (M_1 + M_3)^{-2} \quad (4.29)$$

Similarly, the extreme point on the β axis is given by

$$\beta_{M_i^2} = (M_2 + M_4)^{-2} \quad (4.30)$$

The inverses of these values are then the beginnings of the branch cuts of $A_{\{n_i\nu_i\}}^{\{M_i^2\}}$ in s and t respectively.

Returning to the amplitude $A_{\{n_i\nu_i\}}$ proper, we observe that the branch cuts in s and t move out to the right as any k_i increases. Thus all the cuts are contained in the leading one at $k_i = 0$, $i = 1, \dots, 4$, in which case $M_i^2 \equiv m_i^2 = n_i + |\nu_i|$. and we find that the domain of holomorphy of $A_{\{n_i\nu_i\}}(s, t)$ is given by Fig. 6(b) where

$$(s, t) \in (\mathbf{C} \setminus [(m_1 + m_3)^2, +\infty)) \times (\mathbf{C} \setminus [(m_2 + m_4)^2, +\infty))$$

This completes the proof of Theorem 2.

V. POLES FROM DOUBLE DISPERSION RELATIONS AND STRING DUALITY

Formally, the partial amplitude $A_1(s, t)$ is an infinite sum of ϕ^3 -like box diagrams and as such exhibits double branch cut singularities in both s and t . These branch cuts are clearly displayed by the double dispersion relation derived in the previous section. However,

in §II, we saw that additional pole singularities on top of branch cuts are expected as well, and their appearance must be elucidated.

In the present section, we show that these poles in s and t , lying on top of branch cuts are indeed included in the double dispersion representation, and can be directly extracted from it. In local quantum field theory – with a finite number of fields – poles never arise from dispersion relations (in perturbation theory), and we shall link their appearance to the property of string duality.

Exhibiting simple and double poles on top of branch cuts

We show how the integration in one of the dispersion parameters (say τ), produces a simple spectral density which exhibits double and simple poles (in s) at positive even integers. We represent the amplitude by a simple dispersion relation

$$A_{\{n_i\nu_i\}}(s, t) = \int_0^\infty d\sigma \frac{\rho_{\{n_i\nu_i\}}(s, t; \sigma)}{\sigma - s} \quad (5.1)$$

where the simple spectral density is defined by

$$\rho_{\{n_i\nu_i\}}(s, t; \sigma) = \int_0^\infty d\tau \frac{\rho_{\{n_i\nu_i\}}(s, t; \sigma, \tau)}{\tau - t} \quad (5.2)$$

The double spectral density $\rho_{\{n_i\nu_i\}}(s, t; \sigma, \tau)$ was defined and constructed explicitly in §IV (4.17). The τ -integration may be carried out because $\rho_{\{n_i\nu_i\}}$ depends on τ only through a single φ -function. We make use of a standard relation between the Mellin transform f and the inverse Laplace transform φ (see Appendix §B, in (B.25)) :

$$\int_0^\infty d\tau \frac{1}{\tau - t} \varphi_{n_4\nu_4}(s, u_2\tau - x) = f_{n_4\nu_4}(s, u_2t - x) \quad (5.3)$$

and evaluate $\rho_{\{n_i\nu_i\}}$:

$$\begin{aligned} \rho_{\{n_i\nu_i\}}(s, t; \sigma) &= \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \varphi_{n_1\nu_1}(t, \beta_1) \varphi_{n_2\nu_2}(s, \beta_2) \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 \\ &\quad \times \int_{x_o}^\infty dx (x - x_o)^2 \varphi_{n_3\nu_3}(t, u_1\sigma - x) f_{n_4\nu_4}(s, u_2t - x) \end{aligned} \quad (5.4)$$

From the appearance of f as the last factor in the integrand, it is clear that simple poles in s occur at positive integers. Moreover, there are further simple poles in s appearing from the fact that the β_2 integration now extends to ∞ . We shall now identify these additional double poles more explicitly.

For fixed σ and any values of n_i and ν_i , we have $0 \leq x \leq \sigma$, and $0 \leq \beta_1 \leq \sigma$, so that in an expansion in coefficients $C_k(t)$, both functions $\varphi(t, \cdot)$ produce only a finite number of terms, indeed, of order σ . The only possible origin then of poles as a function of s is from the range $\beta_2 \rightarrow \infty$, which occurs when $u_2 \rightarrow 0$. We see that for fixed σ , $\rho(s, t; \sigma)$ is

an entire function of t . We make use of this analytic behavior in t to expand both $\varphi(t, \cdot)$ functions in a series, as given by (B.18). As a result, we obtain

$$\rho_{\{n_i \nu_i\}}(s, t; \sigma) = \sum_{k_1, k_3=0}^{\infty} C_{k_1}(t) C_{k_1+|\nu_1|}(t) C_{k_3}(t) C_{k_3+|\nu_3|}(t) \rho_{\{n_i \nu_i\}, k_1, k_3}(s, t; \sigma) \quad (5.5)$$

In view of the preceding remarks, the functions $\rho_{\{n_i \nu_i\}, k_1, k_3}(s, t; \sigma)$ vanish whenever k_1 or $k_3 > \sigma$, and the above sum is effectively finite. Furthermore, we have

$$\begin{aligned} \rho_{\{n_i \nu_i\}, k_1, k_3}(s, t; \sigma) &= \int_0^{\infty} d\beta_2 \varphi_{n_2 \nu_2}(s, \beta_2) \int_0^1 du_1 \int_0^{1-u_1} du_2 f_{n_4 \nu_4}(s, M_3^2 - u_1 \sigma + u_2 t) \\ &\quad \times \theta(u_1 \sigma - M_3^2 - x_0) (u_1 \sigma - M_3^2 - x_0)^2 (1 - u_1 - u_2)^2 \end{aligned} \quad (5.6)$$

where we have defined $M_i^2 = 2k_i + n_i + |\nu_i|$ for $i = 1, 3$, and $x_0 = (M_1^2 u_1 + \beta_2 u_2)(1 - u_1 - u_2)^{-1}$. A useful integral representation is obtained by also expanding $f_{n_4 \nu_4}$ in an infinite Taylor series in powers of t . For each term, of order t^p , we must perform the following integral over u_2 , and then over β_2 :

$$\begin{aligned} &\int_0^{\infty} d\beta_2 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^2 u_2^p (u_1 \sigma - m_3^2 - x_0)^2 \\ &= \frac{2p!}{(p+3)!} \left(u_1(1-u_1)\sigma - m_3^2(1-u_1) - m_1^2 u_1 \right)^{p+3} \theta(u_1 \sigma - m_3^2) \\ &\quad \times \int_0^{\infty} d\beta_2 \varphi_{n_2 \nu_2}(s, \beta_2) (u_1 \sigma - m_3^2 + \beta_2)^{-1-p} \\ &= \frac{2}{(p+3)!} (u_1(1-u_1)\sigma - (1-u_1)m_3^2 - u_1 m_1^2)^{p+3} f_{n_2 \nu_2}^{(p)}(s, m_3^2 - u_1 \sigma) \end{aligned} \quad (5.7)$$

Assembling all other factors as well in (5.6), we finally obtain

$$\begin{aligned} \rho_{\{n_i \nu_i\}, k_1, k_3}(s, t; \sigma) &= \sum_{p=0}^{\infty} \frac{2t^p}{p!(p+3)!} \int_0^1 du \theta\left(\sigma - \frac{M_3^2}{u} - \frac{M_1^2}{1-u}\right) \\ &\quad \times (\sigma u(1-u) - (1-u)M_3^2 - uM_1^2)^{p+3} \\ &\quad \times f_{n_2 \nu_2}^{(p)}(s, M_3^2 - u\sigma) f_{n_4 \nu_4}^{(p)}(s, M_3^2 - u\sigma) \end{aligned} \quad (5.8)$$

Here, we use the notation that $f_{n\nu}^{(p)}(s, \alpha) = \partial_{\alpha}^p f_{n\nu}(s, \alpha)$ and this function has simple poles in s at positive integers, strictly greater than p , as can be seen from (B.13) and (B.14). As a result, ρ contains terms holomorphic in s but also terms with single and double poles in s at positive integers. These poles occur in the single spectral density so that after integration over σ , they will be present in the full amplitude on top of a branch cut in s on the positive real axis.

An instructive special case is when $t = 0$, which forces $k_1 = k_3 = 0$ in the sum (5.5) and $p = 0$ in (5.8), so that

$$\begin{aligned} \rho_{\{n_i\nu_i\}}(s, 0; \sigma) = \frac{1}{3}\delta_{\nu_1,0}\delta_{\nu_3,0} \int_0^1 du \, \theta\left(\sigma - \frac{n_3}{u} - \frac{n_1}{1-u}\right) \{ (u(1-u)\sigma - n_3(1-u) - un_1 \}^3 \\ \times f_{n_2\nu_2}(s, n_3 - u\sigma) f_{n_4\nu_4}(s, n_3 - u\sigma) \end{aligned} \quad (5.9)$$

Here double and simple poles in s are clearly exposed.

Connection with String Duality

We have seen in Sections III and IV that, formally, the superstring amplitude is a sum of ϕ^3 -like box diagrams. However, this sum – which in particular goes over all masses of internal propagators – is not uniformly convergent and actually produces poles. Physically, the appearance of poles is related to string duality, as was already the case to tree level. To see this, we write the tree level amplitude as a sum of poles in the s -channel

$$A_o(s, t, u) = \sum_{n=0}^{\infty} \frac{\Gamma(-u/2)\Gamma(t/2 + n + 1)}{u/2\Gamma(1 + s/2)\Gamma(1 + t/2)^2 n!} \frac{1}{-s/2 + n} \quad (5.10)$$

Each term also has poles in u , but not in t . The series is absolutely convergent for $\text{Re}(t) < 0$, and thus we have only poles in s in this region, as well as poles in u , but the series is holomorphic in t . To define the series for $\text{Re}(t) \geq 0$, we must analytically continue it from $\text{Re}(t) < 0$. The analytic continued expression has poles in t this time, which can be seen by simply interchanging s and t in (5.10). Pictorially, this result is represented in Fig. 8. What happens to one loop level is completely analogous. Summations over intermediate masses in the string box diagram are only convergent for $\text{Re}(s), \text{Re}(t) < 0$, and must be analytically continued. In doing so, one produces poles in the crossed channels. This is precisely the meaning of expressions such as (5.1,2) and (5.8) which contain poles. Pictorially, what we have shown is that the sum over box graphs must be analytically continued and that poles are found in the cross channels in the way depicted in Fig. 9. To state that the box diagrams and the triangle graph or the polarization graphs are equal to one another – as is customarily done in the old dual model formulation – is misleading; instead the one loop amplitude contains all contributions at once, after proper analytic continuation.

Cancellation of poles at odd integers

The Mellin transform $f_{n\nu}(s, \alpha)$ is meromorphic in s with simple poles for positive integers. The poles at even positive integers correspond to physical on-shell superstring states, and are expected on physical grounds, as explained in §II. The poles at odd integer values of s are spurious and do not correspond to the propagation of any physical degrees of freedom. In our construction of the analytically continued amplitude, the existence of these poles results from splitting up the full amplitude $A_1(s, t, u)$ into partial amplitudes $A(s, t)$, $A(t, u)$ and $A(u, s)$. This splitting up of the domain is performed in such a way that pole type singularities arising from complex integrals are integrated over only 180 degrees around the singularity, instead of 360 degrees. In fact this phenomenon is identical to and results from the same splitting up problem that arises at tree level.

For example the tree level amplitude $A_0(s, t, u)$ is an integral over the full complex plane as in (2.5), and poles only arise at positive even integers in s, t, u . Now let us split the integration into the one over the unit disk $|z| \leq 1$ and the complementary region $|z| \geq 1$. Each of these integrals is just precisely the Mellin transform f

$$\begin{aligned} A_0(s, t, u) &= -\frac{2}{s^2} \int_{\mathbf{C}} d^2 z |z|^{-2-t} |1-z|^{-s} \\ &= -\frac{4\pi}{s^2} [f_{00}(s, t) + f_{00}(s, u)] \end{aligned} \quad (5.14)$$

It is shown in Appendix B that $f_{00}(s, t)$ has simple poles in t at positive even integers only. Thus the pole structure in t and u is not affected by the splitting up of the z -domain of integration. On the other hand, the splitting into $|z| \leq 1$ and $|z| \geq 1$ precisely cuts the pole region around $z = 1$ in half, and as a result, each half now has additional poles at odd positive integers as well. But in the sum these poles cancel out.

The same Mellin transforms occur in the one-loop analytically continued amplitude, and the poles at odd integers cancel as well once all partial amplitudes have been put together. To see this, one has to make heavy use of the detailed properties of the coefficient functions $P_{n_i \nu_i}(s, t)$, and we shall not provide the check of this cancellation here.

VI. HETEROTIC STRING AMPLITUDES

We consider the one-loop scattering amplitudes for four massless bosonic string states in the heterotic $E_8 \times E_8$ or $\text{Spin}(32)/Z_2$ superstrings [20]. Whereas for Type II superstrings, there was a single massless supermultiplet, for heterotic strings there is a supergravity multiplet as well as a super Yang Mills multiplet. So each S -matrix state can be either one of these. To exemplify the analytic continuation procedure, we shall specialize to a simple subcase: that of the scattering of four massless charged gauge bosons.

The scattering amplitude for four gauge bosons with space-time momenta k_i^μ polarization vectors ϵ_i^μ and root (weight) lattice momenta K_i^I ($i = 1, \dots, 4$; $\mu = 1, \dots, 10$; $I = 1, \dots, 16$) is given by

$$\begin{aligned} A_\ell^H(k_i, \epsilon_i, K_i) &= (2\pi)^{10} \delta(k) g^4 \epsilon_1^{\mu_1}(k_1) \epsilon_2^{\mu_2}(k_2) \epsilon_3^{\mu_3}(k_3) \epsilon_4^{\mu_4}(k_4) \\ &\quad K_{\mu_1 \mu_2 \mu_3 \mu_4}(k_i) A_\ell^H(s, t, u; S, T, U) \end{aligned} \quad (6.1)$$

The kinematical factor K is identical to the one encountered in (2.3), and the root (weight) lattice Mandelstam variables S, T and U as defined by

$$\begin{aligned} S &= -(K_1 + K_2)^2 = -4 - 2K_1 \cdot K_2 \\ T &= -(K_2 + K_3)^2 = -4 - 2K_2 \cdot K_3 \\ U &= -(K_1 + K_3)^2 = -4 - 2K_1 \cdot K_3 \end{aligned} \quad (6.2)$$

Lattice momentum conservation implies

$$\sum_i K_i = 0; \quad S + T + U = -8 \quad (6.3)$$

Since the lattice is self-dual and even, the allowed values for $K_i \cdot K_j$ are 2, 1, 0, -1 or -2, or equivalently S, T, U take on the values -8, -6, -4, -2 or 0. The tree level answer is given by [20]

$$A_0^H(s, t, u; S, T, U) = \pi \frac{\Gamma(-\frac{s+S}{2} - 1) \Gamma(-\frac{t+T}{2} - 1) \Gamma(-\frac{u+U}{2} - 1)}{\Gamma(1 + \frac{s}{2}) \Gamma(1 + \frac{t}{2}) \Gamma(1 + \frac{u}{2})} \quad (6.4)$$

with obvious and physically acceptable analytic continuation, analogous to the Type II case.

The one-loop level reduced scattering amplitude is given by [20]

$$A_1^H(s, t, u; S, T, U) = \int_F \frac{d^2 \tau}{\tau_2^2} \prod_{i=1}^4 \int \frac{d^2 z_i}{\tau_2} \prod_{i>j} e^{\frac{1}{2} s_{ij} G(z_i, z_j)} \overline{\mathcal{L}(K_i, z_i, \tau)} \quad (6.5)$$

Here the Green function G and the Mandelstam variables are those of (2.12) and (2.3) respectively. The function \mathcal{L} is the modification required in going from the Type II to the heterotic case, and is given in terms of the following lattice sum

$$\mathcal{L}(K_i, z_i, \tau) = \eta(\tau)^{-24} \prod_{i<j} \left(\frac{\vartheta_1(z_i - z_j | \tau)}{\vartheta_1'(0 | \tau)} \right)^{K_i \cdot K_j} \vartheta_\Lambda \left(\sum_i K_i z_i | \tau \right) \quad (6.6)$$

$$\vartheta_\Lambda(x^I | \tau) = \sum_{p \in \Lambda} e^{i\pi \tau p \cdot p + 2\pi i p \cdot x}$$

where ϑ_Λ is the Λ - lattice ϑ -function, for Λ the lattices corresponding to $E_8 \times E_8$ or $\text{Spin}(32)/Z_2$. Here the Dedekind function is defined by

$$\eta(\tau) = e^{i\pi \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau n}) \quad (6.7)$$

The above expression may be obtained by straightforwardly translating the expressions in [17], or by direct calculation using chiral splitting techniques, as explained in [2,22]. The latter easily generalizes to the case of other scattering amplitudes as well, and also to higher point functions. Notice that \mathcal{L} is a function only of $K_i^2 = 2$ and $K_i \cdot K_j$, which may all be expressed in terms of S, T and U only.

As in Type II superstring case, we set $z_4 = 0$ by translation invariance, and the remaining integration region may be naturally decomposed according to the orderings of $0 \leq \text{Im}(z_1), \text{Im}(z_2), \text{Im}(z_3) \leq \tau_2$

$$A_1^H(s, t, u; S, T, U) = 2A_1^H(s, t; S, T) + 2A_1^H(t, u; T, U) + 2A_1^H(u, s; U, S) \quad (6.8)$$

where $A_1^H(s, t; S, T)$ is given by (6.5) with the ordering (2.12), and we express $u = -s - t$, $U = -S - T - 8$. We study the function $A_1^H(s, t; S, T)$ which is given by equation (6.5) with the definite ordering

$$0 \leq \text{Im}(z_1) \leq \text{Im}(z_2) \leq \text{Im}(z_3) \leq \tau_2 \quad (6.9)$$

The one loop amplitude for scattering of four gauge bosons takes on a form completely analogous to that of Type II superstring in (3.11) and we have

$$A_1^H(s, t; S, T) = \int_F \frac{d^2\tau}{\tau_2^2} \int_0^{2\pi} \prod_{i=1}^3 \frac{d\alpha_i}{2\pi} \int_0^1 \prod_{i=1}^4 du_i \quad (6.10)$$

$$\times |q|^{-(su_1u_3+tu_2u_4)} \mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T)$$

where we have as a result of (6.5)

$$\mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T) = \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) \overline{\mathcal{R}^H(|q|^{u_i}, \alpha_i; S, T)} \quad (6.11)$$

For the Type II string in (3.16), we employed an expansion in a uniformly and absolutely convergent power series in which we treated exactly four of the ϑ - function factors. The analogous expansion in the heterotic case is given as follows :

$$\mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T) = \prod_{i=1}^4 (1 - e^{i\alpha_i} |q|^{u_i})^{-s_i/2} (1 - e^{-i\alpha_i} |q|^{u_i})^{-s_i/2 - S_i/2 - 2} \quad (6.12)$$

$$\times \sum_{n_i=0}^{\infty} \sum_{\nu_i=-n_i}^{n_i} P_{n_i\nu_i}^{(4)H}(s, t; S, T) \prod_{i=1}^4 |q|^{u_i n_i} e^{i\alpha_i \nu_i}$$

The fundamental hypergeometric functions for the heterotic string case are given by

$$\Phi(s, t; S, T; |q|^{u_i}) = \sum_{n_i=0}^{\infty} \sum_{\nu_i=-n_i}^{n_i} P_{n_i\nu_i}^{(4)H}(s, t; S, T)^H \Phi_{\{n_i\nu_i\}}^H(s, t; S, T; |q|^{u_i}) \quad (6.13)$$

A formula for the expansion coefficients $P_{n_i\nu_i}^{(4)H}$ is contained in Appendix C. The functions $\Phi_{\{n_i\nu_i\}}^H$ are defined by

$$\Phi_{\{n_i\nu_i\}}^H(s, t; S, T; |q|^{u_i}) = \prod_{i=1}^4 \Phi_{n_i\nu_i}(s_i, s_i + S_i + 4; |q|^{u_i})$$

with

$$\Phi_{n\nu}^H(s, S; |q|^u) = \int_0^{2\pi} \frac{d\alpha}{2\pi} |q|^{(n-1)u} e^{i(\nu+1)\alpha} (1 - e^{i\alpha} |q|^u)^{-s/2} (1 - e^{-i\alpha} |q|^u)^{-S/2} \quad (6.14)$$

As before, it is convenient to introduce the Laplace transform $\varphi_{n\nu}^H(s, S; \beta)$ by

$$\Phi_{n\nu}^H(s, S; x^2) = \int_0^\infty d\beta x^\beta \varphi_{n\nu}^H(s, S; \beta) \quad (6.15a)$$

and the Mellin transform $f_{n\nu}^H(s, S; \alpha)$ by

$$f_{n\nu}^H(s, S; \alpha) = \int_0^1 dx x^{-\alpha-1} \Phi_{n\nu}^H(s, S; x^2) \quad (6.15b)$$

In terms of hypergeometric functions, the functions $\Phi_{n\nu}^H(s, S; \beta)$ can be expressed as

$$\Phi_{n\nu}^H(s, S; |q|^u) = C_{|\nu+1|}(s_\nu) F\left(\frac{s}{2} + (\nu+1)_-, \frac{S}{2}; |\nu+1|+1; |q|^{2u}\right) |q|^{(n+|\nu+1|-1)u} \quad (6.16)$$

where s_ν is S or s , and $(\nu+1)_-$ is 0 or $|\nu+1|$, depending respectively on whether $\nu+1$ is non-negative or non-positive. This implies in particular that $f_{n\nu}^H(s, S; \alpha)$ is a meromorphic function with poles in $s+S$ at positive integers. Furthermore, $\varphi_{n\nu}^H(s, S; \beta)$ can be expanded in a series of the form (B.18)

$$\varphi_{n\nu}^H(s, S; \beta) = \sum_{k=0}^{\infty} C_{k,\nu}^H(s, S) \delta(2k + n + |\nu+1| - 1 - \beta) \quad (6.17)$$

with

$$C_{k,\nu}^H(s, S) = C_k(s + 2(\nu+1)_-) C_k(S) C_{k+|\nu+1|}(s_\nu) (C_k(s_\nu + 2|\nu+1|))^{-1} \quad (6.18)$$

Analytic continuations can now be carried out along precisely the same lines as in the case of Type II superstrings. Thus if we set

$$\Psi_{\{n_i\nu_i\}}(s_i, S_i; \beta_i) = \prod_{i=1}^4 \varphi_{n_i\nu_i}^H(s_i, s_i + S_i + 4; \beta)$$

we arrive at

Theorem 4. *a) For any fixed integer N , the scattering amplitude of four massless charged gauge bosons in the heterotic string $A_1^H(s, t)$ can be expressed as*

$$A^H(s, t; S, T) = \sum_{n_1+\dots+n_4 \leq 4N} \sum_{|\nu_i| \leq n_i} P_{\{n_i\nu_i\}}^{(4)H}(s, t; S, T) A_{\{n_i\nu_i\}}^H(s, t) + M_N^H(s, t) \quad (6.19)$$

where $M_N^H(s, t; S, T)$ is a meromorphic function of both s and t in the half-space $\text{Re } s, \text{Re } t < N$, and the partial amplitudes $A_{\{n_i\nu_i\}}^H(s, t)$ are given by double dispersion relations

$$A_{\{n_i\nu_i\}}^H(s, t) = \int_0^\infty d\sigma \int_0^\infty d\tau \left(\frac{\rho_{\{n_i\nu_i\}}^H(s, t; S, T; \sigma, \tau)}{(\sigma - s)(t - \tau)} - \Lambda_{\{n_i\nu_i\}}^H(s, t; S, T; \sigma, \tau) \right) + M_{\{n_i\nu_i\}}^H(s, t; S, T) \quad (6.20)$$

The double spectral density $\rho_{\{n_i\nu_i\}}^H(s, t; S, T; \sigma, \tau)$ is given by the formula (4.17), with $\Psi_{\{n_i\nu_i\}}$ there replaced by the corresponding $\Psi_{\{n_i\nu_i\}}^H$ for the heterotic string. The subtraction terms $\Lambda_{n_i\nu_i}^H$ are all globally meromorphic functions. For each $\{n_i\nu_i\}$, the double dispersion relation in (6.20) defines a holomorphic function of s and t in the complex plane cut along the positive real axis, beginning at $(m_1 + m_3)^2$ and $(m_2 + m_4)^2$ respectively, where

$$m_i^2 = n_i + |\nu_i + 1| - 1 \quad (6.21)$$

b) The above double dispersions can be recast as simple double dispersion relations of the form (5.1), with simple spectral densities $\rho_{\{n_i\nu_i\}}(s, t; S, T; \sigma)$ given by (5.5-5.8), where the expansion coefficients $C_k(s)C_{k+|\nu|}(s)$ and the Mellin transforms $f_{n\nu}(s, \alpha)$ have been replaced respectively by their heterotic analogues $C_{k,\nu}^H(s, S)$ and $f_{n\nu}^H(s, S; \alpha)$. In particular, the simple spectral density $\rho_{\{n_i\nu_i\}}^H(s, t; S, T; \sigma)$ has double poles in $2s + S + 4$ at positive even integers.

c) As illustration, for the forward scattering amplitude $t = 0$ for the heterotic string we have with m_i^2 as in (6.21)

$$\begin{aligned} \rho_{\{n_i\nu_i\}}^H(s, t = 0; S, T; \sigma) = & \theta(\nu_1 + 1)\theta(\nu_3 + 1)C_{\nu_1+1}(T + 4)C_{\nu_3+1}(T + 4) \\ & \times \int_0^1 du \theta\left(\sigma - \frac{m_3^2}{u} - \frac{m_1^2}{1-u}\right) \times (u(1-u)\sigma - m_3^2(1-u) - m_1^2u) \\ & \times f_{n_1\nu_2}^H(s, s + S + 4; m_3^2 - u\sigma) f_{n_4\nu_4}^H(s, s + S + 4; m_3^2 - u\sigma) \end{aligned} \quad (6.22)$$

VII. APPLICATIONS

We shall complement the abstract treatment of the analytic continuation of the one-loop amplitudes in the preceding sections with a discussion of three simple, but very important applications.

The $i\epsilon$ Prescription

The first application follows directly from the existence and explicit forms of the analytic continuation constructed before. It concerns the correct specification in string loop amplitudes of an $i\epsilon$ prescription, familiar from quantum field theory. The rules of quantum field perturbation theory naturally provide each propagator with an $i\epsilon$ prescription, and this prescription uniquely indicates the analytic structure of amplitudes as a function of external momenta. In Appendix D, it is shown how this prescription gives rise to dispersion relations to one loop order in quantum field theory.

In string perturbation theory however, the formulation is not in terms of all the propagators for a given string diagram. In a most standard fashion, no internal momenta are retained, and the amplitude is written directly as an integral over moduli. In this formulation, the introduction of an $i\epsilon$ prescription is obscure. Even in a formulation where

internal loop momenta are retained, all internal propagators are not exhibited, and again the correct introduction of an $i\epsilon$ prescription is obscure.

With the above analytic continuation results however, and especially with their expression in terms of double dispersion relations, the problem of introducing a consistent $i\epsilon$ prescription becomes completely transparent. Indeed, the correct $i\epsilon$ prescription in the amplitude is obtained by defining the following partial amplitudes first:

$$A_{i\epsilon}(s, t) = A_m(s + i\epsilon, t + i\epsilon) + \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho(s, t; \sigma, \tau)}{(s - \sigma + i\epsilon)(t - \tau + i\epsilon)} \quad (7.1)$$

The full amplitude is now gotten by adding the three partial amplitudes:

$$A(s, t, u) = 2A_{i\epsilon}(s, t) + 2A_{i\epsilon}(t, u) + 2A_{i\epsilon}(u, s) . \quad (7.2)$$

In view of the above construction, these amplitudes are finite, with appropriate poles and cuts on the positive real s, t and u axes.

Decay Rates of Massive Strings

The second application concerns special sub parts of the full amplitude. Of particular interest are the residues of the amplitude at double poles in a given channel, say the s -channel. In string theory, as in quantum field theory, the residue at the double pole gives the mass-shift of the respective string states as well as the decay width of the incoming massive string. Massless string states exhibit no mass shifts in superstring models because of the space-time supersymmetry non-renormalization theorems [23].

We note that the mass shift receives contributions from both the meromorphic part of the amplitude as well as from the double dispersion part. The decay width on the other hand receives contributions only from the double dispersion part. This situation arises from the fact that the mass-shift is affected by all (virtual) string intermediate states, whereas the decay width arises from those string states whose mass is less than the original state. As a result, decay widths are easier to calculate from the above construction than the mass shifts. Below we shall present a complete treatment of the partial and total decay widths and we shall present an example of a mass-shift calculation for the lowest massive string state.

We have established in §V that decay widths essentially arise from a sum of quantum field theory square diagrams with all string states moving through the loop. To compute the partial decay width of a string state of mass² = $2N$, into 2 string states of masses squared m_1^2 and m_3^2 , we use the factorization of the superstring 4 point amplitude onto a double pole at $s = 2N$. The t -dependence of the residue may be expanded into Legendre functions indexed by the spin of the massive string states.

The basic formula for the total decay width of a massive string state of mass squared $2N$ is given by

$$\frac{1}{2}\Gamma(2N, t) = \text{Im} \lim_{s \rightarrow 2N} (s - 2N)^2 \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho(s, t; \sigma, \tau)}{(s - \sigma + i\epsilon)(t - \tau)} \quad (7.3)$$

The partial width into string states of mass squared m_1^2 and m_3^2 can easily be deduced from this amplitude by restricting to the relevant terms in $\rho(s, t; \sigma, \tau)$. Note that we included

the three point function square on the left hand side, as this factor must be truncated away from the residue to obtain the decay rate.

We begin by computing the right hand side of (7.3). As we are interested only in the double poles in the s -channel, we do not quite need the full four factor approximation and expansion. In fact we must only retain two factors, namely those responsible for producing the double poles in s . This is valid for $|t| < 2$. Thus, we begin by expanding as follows:

$$\begin{aligned} \int_0^{2\pi} \frac{d\alpha_1}{2\pi} \int_0^{2\pi} \frac{d\alpha_3}{2\pi} \mathcal{I}(|q|^{u_i}, \alpha_i; s, t) \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) &= |1 - e^{i\alpha_2} |q|^{u_2}|^{-s} |1 - e^{i\alpha_3} |q|^{u_3}|^{-s} \\ &\times \sum_{n_i=0}^{\infty} \sum_{\substack{|\nu_2| \leq n_2 \\ |\nu_4| \leq n_4}} P_{n_i; \nu_2, \nu_4}^{(2)}(s, t) |q|^{n_i u_i} e^{i\nu_2 \alpha_2 + i\nu_4 \alpha_4} \end{aligned} \quad (7.4)$$

The coefficient functions $P_{n_i; \nu_2, \nu_4}^{(2)}(s, t)$ are polynomials in s and t , and formulas are given in Appendix C for their evaluation. From the α_2 and α_4 integrals of expansion (7.4) and its subsequent inverse Laplace transform, we derive the *density function*

$$\Psi(s, t; \beta_i) = \sum_{n_i=0}^{\infty} \sum_{\nu_2, \nu_4} P_{n_i; \nu_2, \nu_4}^{(2)}(s, t) \delta(\beta_1 - n_1) \delta(\beta_3 - n_3) \varphi_{n_2 \nu_2}(s; \beta_2) \varphi_{n_4 \nu_4}(s; \beta_4) \quad (7.5)$$

The single spectral density, as discussed in §V for this function Ψ , can be readily deduced along the lines of the calculations in that section. To obtain the partial width into m_1^2 and m_2^2 mass squared string states, we specialize to $n_1 = m_1^2$ and $n_3 = m_3^2$, dropping the sum over n_1 and n_3 . We find, for this partial single spectral density:

$$\begin{aligned} \int_0^{\infty} d\tau \frac{\rho(s, t; \sigma, \tau)}{\tau - t} \Big|_{m_1^2, m_3^2} &= \sum_{\substack{n_2, n_4=0 \\ \nu_2, \nu_4}}^{\infty} P_{n_i; \nu_2, \nu_4}^{(2)}(s, t) \sum_{p=0}^{\infty} \frac{2t^p}{p!(p+3)!} \\ &\times \int_0^1 du f_{n_2 \nu_2}^{(p)}(s; m_3^2 - u\sigma) f_{n_4 \nu_4}^{(p)}(s; m_3^2 - u\sigma) \\ &\times \theta\left(\sigma - \frac{m_3^2}{u} - \frac{m_1^2}{1-u}\right) (\sigma u(1-u) - (1-u)m_3^2 - um_1^2)^{p+3} \end{aligned} \quad (7.6)$$

The double pole part at $s = 2N$ is easily extracted by using the following residue formula derived in Appendix B :

$$\lim_{s \rightarrow 2N} (s - 2N) f_{n\nu}(s, \alpha) = F_{\nu}(2N, \alpha - n) \quad (7.7)$$

where the residue is a polynomial in α of degree $2N - 2$

$$F_{\nu}(2N, \alpha) = -\frac{1}{2\Gamma(N)^2} \prod_{k=1}^{N-1} \left\{k + \frac{\alpha + \nu}{2}\right\} \left\{k + \frac{\alpha - \nu}{2}\right\} \quad (7.8)$$

Furthermore, the 3-point function in (7.3) can be read off from (2.6), so we obtain

Theorem 5. *The partial decay width of a string state of mass $2N$ into two string states of masses m_1^2 and m_3^2 is given by*

$$\begin{aligned} \frac{1}{2}\Gamma(2N, t) \Big|_{m_1^2, m_3^2} \frac{8\pi}{t^2} C_N(t)^2 &= \sum_{\substack{n_2, n_4=0 \\ \nu_2, \nu_4}}^{\infty} P_{m_1^2, n_2, m_3^2, n_4; \nu_2, \nu_4}^{(2)}(2N, t) \sum_{p=0}^{\infty} \frac{2t^p}{p!(p+3)!} \\ &\times \int_0^1 du \times \theta\left(2N - \frac{m_3^2}{u} - \frac{m_1^2}{1-u}\right) \\ &\times (2Nu(1-u) - (1-u)m_3^2 - um_1^2)^{p+3} \\ &\times F_{\nu_2}^{(p)}(2N, m_3^2 - 2Nu - n_2) F_{\nu_4}^{(p)}(2N, m_3^2 - 2Nu - n_4) \end{aligned} \quad (7.9)$$

As $F_{\nu}(2N, \alpha)$ is a polynomial of degree $2N - 2$, only terms with $p \leq 2N - 2$ contribute, and the p sum is finite for fixed N .

Mass Shifts to One Loop

The mass shifts to one-loop are given by the real parts of the residues of $A_1(s, 0)$ at the double poles at $s = 2N$. Unlike the decay rates, the real part incorporates also the contributions of the terms we have set aside as globally meromorphic. In principle, it is possible, although exceedingly tedious, to work out the residues of the globally meromorphic terms at any double pole and sum them up. Thus it is more efficient to proceed in the manner we describe next.

For $t = 0$ the amplitude $A(s, t, u)$ is given by the expression

$$A(s, t, u)|_{t=0} = \frac{1}{2} \int_F \frac{d^2\tau}{\tau_2^2} \int \prod_{i=1}^4 d^2z_i \exp \frac{s}{2} (G(z_1, z_2) + G(z_3, z_4) - G(z_1, z_3) - G(z_2, z_4)) \quad (7.10)$$

The double poles arise from the integral

$$\int \frac{d^2z_2}{\tau_2} \exp\left(\frac{s}{2}G(z_1, z_2)\right) \exp\left(-\frac{s}{2}G(z_2, z_4)\right) \times \int \frac{d^2z_3}{\tau_2} \exp\left(\frac{s}{2}G(z_3, z_4)\right) \exp\left(-\frac{s}{2}G(z_1, z_3)\right) \quad (7.11)$$

over the region when z_2 and z_3 are close respectively to z_1 and z_4 . In this region, $\exp(-G(z_1, z_2)) \sim |z_1 - z_2|^2$ and $\exp(-G(z_3, z_4)) \sim |z_3 - z_4|^2$. In view of the arguments leading to Lemma B.2 in Appendix B, each of the two factors in (7.8) is a globally meromorphic function of s , with poles at s integer ≥ 2 . The residues at $s = 2$ are given respectively by $\exp(-G(z_2, z_4))|_{z_2=z_1}$ and $\exp(-G(z_1, z_3))|_{z_3=z_4}$ and thus are both equal to $\exp(-G(z_1, z_4))$. Integrating now with respect to z_1 and z_4 , we find in view of translation invariance that the residue (i.e. the coefficient of the most singular term in the Laurent expansion) of the forward scattering amplitude $A_1(s, t)|_{t=0}$ at the double pole $s = 2$ is given by

$$Res|_{s=2} = A(-4)$$

where the function $A(s)$ has been defined by

$$A(s) = \int_F \frac{d^2\tau}{\tau_2^2} \int \frac{d^2z}{\tau_2} \exp\left(\frac{s}{2}G(z, 0)\right) \quad (7.12)$$

The integral (7.12) converges only when $0 \leq \text{Re } s < 2$, and thus $A(-4)$ has to be understood in the sense of analytic continuation. The methods we developed earlier readily produce this analytic continuation. In fact, we have

Theorem 6. *a) The function $A(s)$ can be extended to a meromorphic function of s in the complex plane with a cut along the negative real axis. All the poles of $A(s)$ are simple, and can occur only at positive integers starting from $s = 2$;*
b) More precisely, for any fixed positive even integer N , $A(s)$ can be written as

$$A(s) = s \ln s + \sum_{k=1}^{N/2-1} a_k(s) \int_{2k}^{\infty} \frac{d\alpha}{\alpha^2} \ln(s + \alpha) + A_N(s) \quad (7.13)$$

where $A_N(s)$ are holomorphic in the domain $-N < \text{Re } s < 2$, and the coefficients $a_k(s)$ are polynomials in s .

c) The holomorphic terms $A_N(s)$ in (7.13) can be expressed in terms of convergent integrals. In particular

$$\begin{aligned} 16\pi^4 A(-4) = & \int_0^1 du L_2(-2u(1-u)) + \int_{-1/2}^{1/2} d\tau_1 \int_{\sqrt{1-\tau_1^2}}^1 \int_0^1 dy e^{4\pi\tau_2 y(1-y)} \\ & + \int_F \frac{d^2\tau}{\tau_2^2} \int \frac{d^2z}{\tau_2} (\exp(-2G(z, 0)) - e^{4\pi\tau_2 y(1-y)}) \end{aligned} \quad (7.14)$$

where the function $L_2(s)$ is defined by

$$L_2(s) = s \ln s + \int_0^s \frac{dx}{x} (e^{-x} - 1) + e^{-s} - \left(\int_0^\infty \frac{dx}{x} e^{-x} + \int_0^1 \frac{dx}{x} (e^{-x} - 1) \right) s \quad (7.15)$$

Since (a) and (b) of Theorem 6 can be established along the same lines as Theorems 1-4, we discuss only (c). We consider separately the contributions of the two regions $\text{Im } \tau_2 \leq 1$ and $\text{Im } \tau_2 > 1$ of moduli space. The first gives an absolutely convergent integral. The second may be reexpressed as

$$\left(\frac{1}{4\pi^2}\right)^s \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 du |q|^{-su(1-u)} \Phi(s, |q|^{2u}, |q|^{2(1-u)})|_{s=2} \quad (7.16)$$

where the function $\Phi(s, |q|^{2u}, |q|^{2(1-u)})$ is the following version of hypergeometric functions

$$\Phi(s, |q|^{2u}, |q|^{2(1-u)}) = \int_{-1/2}^{1/2} d\tau_1 \int_0^1 dx \left| \prod_{n=0}^{\infty} (1 - q^n e^{2\pi iz})(1 - q^{n+1} e^{-2\pi iz}) \right|^s \quad (7.17)$$

Comparing with (B.4), it is easy to see that the poles of Φ now occur in s at *negative* integer values. In particular $\Phi(s, |q|^{2u}, |q|^{2(1-u)})$ is holomorphic at $s = 2$. Its radius of convergence in each of the variables $|q|^{2u}$ and $|q|^{2(1-u)}$ is 1. By symmetry we may restrict our considerations to $u \leq 1 - u$. We observe now that the integral (7.16) over the region $\tau_2 u \leq 1$ is convergent, since there the factor $|q|^{-2u(1-u)}$ remains bounded. In the complementary region $\tau_2 u \geq 1$, we have $|q|^{2(1-u)} \leq |q|^{2u} \leq e^{-2\pi}$. Since these values lie well within the radius of convergence of Φ , it follows that the expression

$$|q|^{-su(1-u)} [\Phi(s, |q|^{2u}, |q|^{2(1-u)}) - 1] \big|_{s=2}$$

remains bounded. This reduces the analytic continuation of (7.13) to that of the simple expression

$$\left(\frac{1}{4\pi^2}\right) \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 du |q|^{-su(1-u)} \big|_{s=2}$$

This can be done explicitly in terms of the function $L_2(s)$, (see Appendix F) giving

$$\left(\frac{1}{4\pi^2}\right)^2 \int_0^1 du L_2(-2u(1-u))$$

Collecting all the terms gives the expression (7.14).

Appendix A: MEROMORPHIC ANALYTIC CONTINUATIONS

In this appendix, we gather the analytic continuations leading only to meromorphic functions. The simplest is provided by the following lemma, the proof of which was given in the discussion of tree-level string amplitudes in §II, see (2.9)

Lemma A.1. *Let $f(x)$ be any smooth function of x on the interval $[0,1]$. Then the integral*

$$\int_0^1 dx x^{-1-s} f(x)$$

can be extended as a meromorphic function of s in the complex plane, with simple poles at $s = n$, $n = 0, 1, 2, \dots$ and residues $-f^{(n)}(0)/n!$.

Lemma A.2 *Let $\phi(x, y)$ be a smooth function of (x, y) which does not vanish in the unit square $0 \leq x, y \leq 1$. Then for any function $E(x, y, s)$ smooth in x and y and holomorphic in s , the integral*

$$\int_0^1 \int_0^1 dx dy (x^2 + y^2 \phi^2)^{-s/2} E(x, y, s)$$

can be extended as a meromorphic function in the whole s -plane, with poles at $s = 2, 3, 4, \dots$.

Proof. The basic idea, applying here and elsewhere in presence of several different scales, here x and y , is to order them in increasing order, and integrate them successively, beginning with the smallest. More precisely, we write

$$\int_0^1 \int_0^1 dx dy = \int_0^1 dx \int_0^x dy + \int_0^1 dy \int_0^y dx$$

In the first region on the right hand side, we change variables $y \equiv xz$, obtaining

$$\int_0^1 dx x^{1-s} \left\{ \int_0^1 dz (1 + z^2 \phi^2)^{-s/2} E(x, zx, s) \right\}$$

The integral between brackets produces a C^∞ function of x . Lemma A.1 thus applies, showing that the dx integral produces in turn a meromorphic function of s in the whole plane, with poles at $s = 2, 3, 4, \dots$. In the second region, we integrate first with respect to x , using the change of variables $x \equiv yz$, $0 \leq z \leq 1$. The proof of Lemma A.2 is complete.

Lemma A.3. *Let $E(w_i)$ be smooth functions of w_i in the disks $|w_i| \leq 1$, $i = 1, 2, 3$. Then the integrals*

(i)

$$\int_{\frac{1}{2} \leq |w_1| \leq 1} d^2 w_1 |1 - w_1|^{-s} E(w_1)$$

(ii)

$$\int_{\frac{1}{2} \leq |w_2| \leq |w_1| \leq 1} d^2 w_1 d^2 w_2 |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_1 w_2|^{s+t} E(w_1, w_2)$$

(iii)

$$\int_{\substack{\frac{1}{2} \leq |w_i| \leq 1 \\ |w_{i+1}| \geq |w_i|}} \prod_{i=1}^3 d^2 w_i |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_3|^{-t} |1 - w_1 w_2|^{s+t} \\ \times |1 - w_2 w_3|^{s+t} |1 - w_1 w_2 w_3|^{-s} E(w_1, w_2, w_3)$$

can be analytically continued as meromorphic functions of s and t in the whole complex plane. They admit poles of order at most two, in s , t , and $u = -(s+t)$ at integers greater or equal to 2.

Proof. For (i), we introduce the variables $x_1 \equiv 2(1 - |w_1|)$ and $y_1 \equiv \frac{\alpha_1}{\pi}$ and observe that $|1 - w_1|$ can be written as

$$|1 - w_1| = \frac{1}{2} (x_1^2 + y_1^2 \phi^2(x_1, y_1))^{-1/2} \quad \text{for } 0 \leq y_1 \leq 1 \quad (A.1)$$

with a smooth function $\phi(x_1, y_1)$ bounded away from 0 for $\frac{1}{2} \leq |w_1| \leq 1$. In this range, the factor $E(w_1)$ is also a smooth function of x_1 and y_1 . Thus the integral in (i) can be rewritten as

$$\int_0^1 \int_0^1 dx_1 dy_1 (x_1^2 + y_1^2 \phi^2)^{-s/2} E_1(x_1, y_1, s)$$

for another function E_1 still smooth in x_1, y_1 and holomorphic in s . Part (i) follows at once from Lemma A.2.

To deal with (ii) and (iii), there are more scales, and we refine the arguments of Lemma A.2 as follows.

First by exploiting periodicity in the α_i 's ($w_i \equiv e^{i\alpha_i} |w_i|$), we can reduce the study of (ii) to that of an expression of the form

$$\int_{1/2}^1 d|w_i| \int_0^\pi d\alpha_i |1 - e^{i\alpha_1} |w_1||^{-s} |1 - e^{i\alpha_2} |w_2||^{-t} |1 - e^{i(\alpha_1 \pm \alpha_2)} |w_1| |w_2||^{(s+t)/2} E(w_1, w_2) \quad (A.2)$$

Consider first the simpler case where it is $(\alpha_1 - \alpha_2)$ which appears in (A.2). We again order the angles as either $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$, and consider for example the region with

$$0 \leq \alpha_1 \leq \alpha_2 \leq \pi$$

We set then

$$\begin{aligned} 0 &\leq y_1 \equiv \alpha_1 / \pi \leq 1 \\ 0 &\leq y_2 \equiv (\alpha_2 - \alpha_1) / \pi \leq 1 \\ x_i &\equiv 1 - |w_i| \end{aligned}$$

and exploit the formula (A.1) in the range where all angles πy_i are less than π to write (A.2) as

$$\int_0^{1/2} dx_1 \int_0^{x_1} dx_2 \int_0^1 dy_1 \int_0^{1-y_1} dy_2 (x_1^2 + y_1^2 \phi_1^2)^{-s/2} (x_2^2 + (y_1 + y_2)^2 \phi_2^2)^{-t/2} (x_1^2 + x_2^2 + y_2^2 \phi_3^2)^{(s+t)/2} E_1(x_1, x_2, y_1, y_2) \quad (A.3)$$

Here $E_1, \phi_1, \phi_2, \phi_3$ are all smooth functions, with ϕ_1, ϕ_2, ϕ_3 bounded away from 0. The integral (A.3) can now be treated by integrating successive scales. Thus the scaling $x_2 \rightarrow x_1 x_2$ transforms (A.3) into

$$\int_0^{1/2} dx_1 x_1 \int_0^1 dx_2 \int_0^1 dy_1 \int_0^{1-y_1} dy_2 (x_1^2 + y_1^2 \phi_1^2)^{-s/2} (x_1^2 x_2^2 + (y_1 + y_2)^2 \phi_2^2)^{-t/2} \times (x_1^2 + y_2^2 \phi_4^2)^{(s+t)/2} E_2(x_1, x_2, y_1, y_2)$$

where E_2, ϕ_4 are still smooth functions, with ϕ_4 bounded away from 0. If we set $y_i = y k_i$, $y = y_1 + y_2$, and replace the $dy_1 dy_2$ integral by a $dy dk_1 dk_2$ integral, the integrand becomes

$$\delta(1 - k_1 - k_2) (x_1^2 + y^2 k_1^2 \phi_1^2)^{-s/2} (x_1^2 x_2^2 + y^2 \phi_2^2)^{-t/2} (x_1^2 + y^2 k_2^2 \phi_4^2)^{(s+t)/2} E_2$$

This can be now finished off by comparing the x_1 and y_1 scales

$$\int_0^{1/2} dx_1 \int_0^1 dy = \int_0^{1/2} dx_1 \int_0^{2x_1} dy + \int_0^1 dy \int_0^{\frac{1}{2}y} dx_1$$

These two terms lead respectively to

$$\int_0^{1/2} dx_1 x_1 \int_0^1 dx_2 \int_0^1 dy \int_0^1 dk_1 \int_0^1 dk_2 \delta(1 - k_1 - k_2) \times (1 + y^2 k_1^2 \phi_1^2)^{-s/2} (x_2^2 + y^2 \phi_2^2)^{-t/2} (1 + y^2 k_2^2 \phi_4^2)^{s+t} E_2 \quad (A.4)$$

and

$$\int_0^1 dy y^2 \int_0^{1/2} dx_1 x_1 \int_0^1 dx_2 \int_0^1 dk_1 \int_0^1 dk_2 \delta(1 - k_1 - k_2) \times (x_1^2 + k_1^2 \phi_1^2)^{-s/2} (x_1^2 x_2^2 + \phi_2^2)^{-t/2} (x_1^2 + k_2^2 \phi_4^2)^{s+t} E_2 \quad (A.5)$$

In (A.4) only the factor $(x_2^2 + y^2 \phi_2^2)^{-t/2}$ can create singularities. In view of Lemma A.2, it produces a meromorphic function of t , with simple poles at most at $t = 2, 3, 4, \dots$.

In (A.5) we view the dk_i integrals as

$$\int_0^1 \int_0^1 dk_1 dk_2 \delta(1 - k_1 - k_2) = \int_0^{1/2} dk_1 \Big|_{k_2=1-k_1} + \int_0^{1/2} dk_2 \Big|_{k_1=1-k_2}$$

Thus of the three factors in (A.5), only one can contribute singularities. In the range $0 \leq k_1 \leq 1/2$ it is the factor $(x_1^2 + k_1^2 \varphi_1^2)^{-s/2}$, which produces a meromorphic function of s with simple poles in s at $s = 2, 3, 4, \dots$. In the range $0 \leq k_2 \leq 1/2$, the factor $(x_1^2 + k_2^2 \varphi_1^2)^{(s+t)}$ produces simple poles in u ($= -(s+t)$) at $2, 3, 4, \dots$. This completes the treatment of (A.2) when the expression $\alpha_1 - \alpha_2$ appears.

We turn now to the case with $\alpha_1 + \alpha_2$, and break the $d\alpha_1 d\alpha_2$ region of integration further into four regions, defined by $\alpha_i \leq \pi/2$ or $\alpha_i \geq \pi/2$. In three regions, the above arguments apply essentially unmodified. In the last, $\frac{\pi}{2} \leq \alpha_i \leq \pi$, we write the factor with $\alpha_1 + \alpha_2$ as

$$|1 - e^{i(\alpha_1 + \alpha_2)} \lambda_1 \lambda_2|^{s+t} = (x_1^2 + x_2^2 + (1 - \frac{\alpha_1 + \alpha_2}{2\pi})^2 \phi_3^2)^{(s+t)/2}$$

Obviously the other factors are bounded away from 0. Letting $\beta_i = \pi - \alpha_i$ we obtain

$$|1 - e^{i(\alpha_1 + \alpha_2)} \lambda_1 \lambda_2|^{s+t} = (x_1^2 + x_2^2 + (\beta_1 + \beta_2)^2 \phi_4^2)^{(s+t)/2}$$

with $0 \leq \beta_i \leq \frac{\pi}{2}$ and ϕ_4 bounded away from 0. An easy adaptation of Lemma A.2 shows at once that this term is meromorphic in u , with simple poles at $u = 4, 5, 6, \dots$. This establishes (ii).

Although more tedious because of the presence of a greater number of scales, the proof of (iii) is exactly along the same lines. This completes our discussion of Lemma A.2.

The above scaling arguments also yield easily the following version, which we need to handle the error terms in the derivation of the double dispersion relations:

Lemma A.4. *Let $E(\lambda, y)$ and $\phi(\lambda, y)$ be smooth functions of λ and y in the range $0 \leq \lambda_i, y \leq 1$. Then the integral*

$$\int_0^1 \prod_{i=0}^M d\lambda_i \lambda_i^{\ell_i} \int_0^1 dy \left(\prod_{i=0}^M \lambda_i^2 + y^2 \varphi^2 \right)^{-s/2} y^q E(\lambda, y) \quad (A.6)$$

can be analytically continued as a meromorphic function in the s -plane. In fact, for any fixed positive integer N , there exist entire functions $c_{n_0 n_1 \dots n_M}(s)$ so that it can be expressed as

$$\sum_{n_i = q+2+\ell_i}^N c_{n_0 n_1 \dots n_M}(s) \prod_{i=0}^M (s - n_i)^{-1}$$

up to a holomorphic function in $\text{Re } s < N$.

We can discuss now meromorphic continuations as they appear in the reduction process stated in Theorem 1.

Lemma A.5. *Let $\mathcal{I}(u, \alpha, q)$ be as in (3.26), and let $E(u, \alpha, q)$ be any smooth function of u, α , and q . Dependence on s and t is implicit in these functions and is assumed to be holomorphic. Set*

$$\mathcal{E}(s, t, q) \equiv \int_0^{2\pi} \prod_{i=1}^3 \frac{d\alpha_i}{2\pi} \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) \mathcal{I}(|q|^{u_i}, \alpha_i, q) E(|q|^{u_i}, \alpha_i, q) \quad (\text{A.7})$$

Then \mathcal{E} can be analytically continued as a meromorphic function in the whole plane in both s and t , with at most poles in s, t and $-(s+t)$ at integers greater or equal to 2. The coefficients of a pole in one variable are entire in the other variable. Furthermore, \mathcal{E} is uniformly bounded as $\tau_2 \rightarrow \infty$ on any compact set K in s and t away from the poles. Near the poles, the coefficients of the Laurent expansion as well as the holomorphic parts are also uniformly bounded as $\tau_2 \rightarrow \infty$. The bounds depend only on K and on bounds for a finite number of derivatives of $E(|q|^{u_i}, \alpha_i, q)$ of the form

$$\sup \tau_2^{-a} |\partial_u^a \partial_\alpha^b E(|q|^{u_i}, \alpha_i, q)|$$

Proof. The region of integration in the integral (A.7) can be divided into 2^4 subregions by ordering the u_i 's. Consider for example the region

$$0 \leq u_1 \leq u_2 \leq u_3 \leq u_4$$

We choose then u_1, u_2, u_3 as independent variables, and enforce the $\delta(1 - \sum_{i=1}^4 u_i)$ constraint by setting $u_4 = 1 - (u_1 + u_2 + u_3)$. Since $u_4 \geq \frac{1}{4}$ in this region, all factors in $\mathcal{I}(|q|^{u_i}, \alpha_i, q)$ with $|q|^{2u_4}$ are C^∞ and bounded away from 0. They can all be absorbed in a single factor $E(|q|^{u_i}, \alpha_i, q)$ which is C^∞ and bounded away from 0. Thus $\mathcal{E}(s, t, q)$ can be expressed as

$$\begin{aligned} \int_0^{2\pi} \prod_{i=1}^3 \frac{d\alpha_i}{2\pi} \int \prod_{i=1}^3 du_i & |1 - e^{i\alpha_1} |q|^{u_1}|^{-t} |1 - e^{i(\alpha_1 + \alpha_2)} |q|^{u_1 + u_2}|^{s+t} \\ & \times |1 - e^{i\alpha_2} |q|^{u_2}|^{-s} |1 - e^{i\alpha_3} |q|^{u_3}|^{-t} |1 - e^{i(\alpha_2 + \alpha_3)} |q|^{u_2 + u_3}|^{s+t} \\ & \times |1 - e^{i(\alpha_1 + \alpha_2 + \alpha_3)} |q|^{u_1 + u_2 + u_3}|^{-s} E(|q|^{u_i}, \alpha_i, q) \end{aligned}$$

with the $\prod du_i$ integral over the region $0 \leq u_1 \leq u_2 \leq u_3, u_1 + u_2 + 2u_3 \leq 1$. If we again introduce $w_i \equiv e^{i\alpha_i} |q|^{u_i}$, the range spanned by $|w_1|$ is $|q| \leq |w_i| \leq 1$. We note however that analytic continuation is only needed when $|w_i|$ is close to 1, say $\frac{1}{2} \leq |w_i| \leq 1$. It is necessary to separate out the regions where $|q| \leq |w_i| \leq \frac{1}{2}$, and not change to the new variable w_i there, since the Jacobian of that change introduces negative powers of $|q|$ which blow up as $|q| \rightarrow 0$. Thus we break up the region $\{0 \leq u_1 \leq u_2 \leq u_3, u_1 + u_2 + 2u_3 \leq 1\}$

further into

$$\begin{aligned}
& \{|q| \leq |w_3| \leq |w_2| \leq |w_1| \leq 1; |w_1||w_2||w_3|^2 \geq |q|\} \\
&= \{|q| \leq |w_3| \leq |w_2| \leq |w_1| \leq \frac{1}{2}; |w_1||w_2||w_3|^2 \geq |q|\} \\
&\quad \cup \{|q| \leq |w_3| \leq |w_2| \leq \frac{1}{2} \leq |w_1|; |w_1||w_2||w_3|^2 \geq |q|\} \\
&\quad \cup \{|q| \leq |w_3| \leq \frac{1}{2} \leq |w_2| \leq |w_1|; |w_1||w_2||w_3|^2 \geq |q|\} \\
&\quad \cup \{|q| \leq \frac{1}{2} \leq |w_3| \leq |w_2| \leq |w_1|; |w_1||w_2||w_3|^2 \geq |q|\}
\end{aligned}$$

The first region obviously produces an entire function of s and t . The contribution of the next region can be expressed as

$$\begin{aligned}
& \int_0^{2\pi} \prod_{i=1}^3 \frac{d\alpha_i}{2\pi} \int \prod_{i=1}^3 du_i |1 - e^{i\alpha_1} |q|^{u_1}|^{-t} E_1(|q|^{u_i}, \alpha_i, q) \\
&= \int_{\frac{1}{2} \leq |w_1| \leq 1} d^2 w_1 |1 - w_1|^{-1} \left[\int \prod_{i=1}^2 \frac{d\alpha_i du_i}{2\pi} |w_1|^{-1} E_1(w_1, |q|^{u_2}, |q|^{u_3}, \alpha_2, \alpha_3, q) \right]
\end{aligned}$$

The integral between brackets is over the region

$$u_i \geq (2\pi\tau_2)^{-1} \ln 2, \quad u_2 + 2u_3 \leq 1 + \tau_2^{-1} \ln |w_1|$$

It produces a smooth function of w_1 , bounded together with its derivatives in the range $\frac{1}{2} \leq |w_1| \leq 1$, uniformly as $\tau_2 \rightarrow \infty$. Part (i) of Lemma A.3 gives then the desired statement.

In the same way, the last two regions give rise to integrals of the form

$$\begin{aligned}
& \int \int_{\frac{1}{2} \leq |w_i| \leq 1} d^2 w_1 d^2 w_2 |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_1 w_2|^{s+t} \\
& \quad \times \left[\int \int \frac{d\alpha_3 du_3}{2\pi} |w_1|^{-1} |w_2|^{-1} E_2(w_1, w_2, |q|^{u_3}, \alpha_3, q) \right]
\end{aligned}$$

where the integral between brackets is over the region

$$(\pi\tau_2)^{-1} \ln 2 \leq 2u_3 \leq 1 + \tau_2^{-1} (\ln |w_1| + \ln |w_2|)$$

and of the form

$$\begin{aligned}
& \int \int \int_{\frac{1}{2} \leq |w_i| \leq 1} d^2 w_1 d^2 w_2 d^2 w_3 |1 - w_1|^{-t} |1 - w_2|^{-s} |1 - w_3|^{-t} |1 - w_1 w_2|^{s+t} \\
& \quad \times |1 - w_2 w_3|^{s+t} |1 - w_1 w_2 w_3|^{-s} E_3(w, q)
\end{aligned}$$

Applying respectively (ii) and (iii) of Lemma A.3 gives Lemma A.5.

The bounds formulated in Lemma A.5 allow us to determine when we can integrate in τ and still have meromorphic functions. The case of interest to us is covered by the following:

Lemma A.6. *Let \mathcal{I} be as in Lemma A.5 and let $E(|q|^{u_i}, \alpha_i, q)$ be a smooth function of u_i , α_i , and q which is bounded together with all its derivatives as $q \rightarrow 0$. Then the integral*

$$\int \frac{d^2\tau}{\tau_2^2} \int \prod_{i=1}^4 \frac{d\alpha_i}{2\pi} \delta(2\pi\tau_1 - \sum_{i=1}^4 \alpha_i) \int \prod_{i=1}^4 du_i \delta\left(1 - \sum_{i=1}^4 u_i\right) |q|^{-(su_1u_2+tu_2u_4)} \\ \times \mathcal{I}(|q|^{u_i}, \alpha_i, q) E(|q|^{u_i}, \alpha_i, q)$$

over the region $D \cap \{\tau_2(u_1 + u_2) \leq 1\}$ can be analytically continued as a meromorphic function of both s and t in the whole plane. It has poles at most in s , t , and $u = -(s + t)$ at integers, with residues entire in the remaining variables.

Proof. First we note that Lemma A.5 still holds when the region of integration in $\prod_{i=1}^4 du_i$ is modified to $\tau_2(u_1 + u_2) < 1$ in the integral (A.7), with a few simple modifications of the same proof. Next, we need to show that the integral $d\tau^2/\tau_2^2$ over $D \cap \{\tau_2(u_1 + u_2) < 1\}$ of the analytic continuations obtained through Lemma A.5 for each fixed τ is finite. For each τ , Lemma A.5 shows that the integral is a meromorphic function in the whole plane with the indicated poles. Furthermore, the statement (c) reduces estimating the size of the integral to estimating the size of

$$\prod_{i=1}^4 \tau_2^{-a_i} \left(\frac{\partial}{\partial u_i} \right)^{a_i} |q|^{-su_1u_2-tu_2u_4}$$

in the region where $\tau_2 u_i \leq 1$, and of $|q|^{-(su_1u_2+tu_2u_4)}$ in the domain of integration $\tau_2(u_1 + u_2) \leq 1$. These are uniformly bounded as $\tau_2 \rightarrow \infty$, and thus the integral over $D \cap \{\tau_2(u_1 + u_2) < 1\}$ can be carried out, giving the desired result.

Appendix B : HYPERGEOMETRIC FUNCTIONS

We define binomial coefficients by the relation

$$(1 - z)^{-s/2} = \sum_{k=0}^{\infty} C_k(s) z^k \quad C_k(s) = \frac{\Gamma(k + s/2)}{\Gamma(s/2)\Gamma(k + 1)} \quad (B.1)$$

The hypergeometric function ${}_2F_1 = F$ may be defined by the absolutely convergent series for $|z| < 1$:

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b + k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c + k) k!} z^k \quad (B.2)$$

For $|z| \geq 1$, it can be defined by analytic continuation, with a branch cut along $[1, \infty]$. As a function of a , b and c , the function is meromorphic. F satisfies the famous Gauss hypergeometric differential equation, which we shall not need here. F also satisfies an important “reciprocity” relation

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b+1-c; 1-z) \\ + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z) \quad (B.3)$$

In particular F and its derivatives at $z = 1$ are meromorphic functions of a , b , and c . *Mellin transformation of hypergeometric functions*

In our discussion of the Type II superstring, we encountered a special case of the hypergeometric function

$$z^{n+|\nu|} C_{|\nu|}(s) F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; z^2\right) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \left|1 - z e^{i\alpha}\right|^{-s} e^{i\nu\alpha} z^n \quad (B.4)$$

For the heterotic string, the corresponding integral was given in §VII. This expression further generalizes (B.4), but in fact since S , T , U are negative even integers, the expressions for the heterotic string are simply related to (B.4). The Mellin transform of the special hypergeometric function of (B.4) is defined by

$$f_{n\nu}(s, \alpha) = C_{|\nu|}(s) \int_0^1 dx \ x^{-\alpha-1+n+|\nu|} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) \quad (B.5)$$

Note that we have $f_{n\nu}(s, \alpha) = f_{0\nu}(s, \alpha - n)$, however, it is convenient to exhibit the n -dependence explicitly in our string formulas. For $\text{Re}(\alpha) < 0$, and $\text{Re}(s) < 2$, we may write an absolutely convergent series expansion

$$f_{n\nu}(s, \alpha) = \sum_{k=0}^{\infty} C_k(s) C_{k+|\nu|}(s) \frac{1}{2k + n + |\nu| - \alpha} \quad (B.6)$$

which allows us to identify this functions with a higher hypergeometric function evaluated at unity:

$$f_{n\nu}(s, \alpha) = \frac{C_{|\nu|}(s)}{n + |\nu| - \alpha} {}_3F_2\left(\frac{s}{2}, \frac{s}{2} + |\nu|, \frac{n+|\nu|-\alpha}{2}; 1 + |\nu|, 1 + \frac{n+|\nu|-\alpha}{2}; 1\right) \quad (B.7)$$

In string theory, this function arises from the following integral representation

$$f_{n\nu}(s, t) = \frac{1}{2\pi} \int_{|z| \leq 1} d^2 z \ |z|^{-t-2} |1 - z|^{-s} z^{\frac{1}{2}(n+\nu)} \bar{z}^{\frac{1}{2}(n-\nu)} \quad (B.8)$$

The integral extended over the full complex plane equals a ratio of Γ -functions, related to the Virasoro-Shapiro amplitude in string theory. Splitting the complex plane into the

regions inside and outside the unit circle results in two integrals of the type (B.8), whose sum is again a ratio of Γ functions :

$$f_{0\nu}(s, t) + f_{0\nu}(s, u) = -\frac{s^2}{4} \frac{\Gamma(-s/2)\Gamma((-t+\nu)/2)\Gamma((-u+\nu)/2)}{\Gamma(1+s/2)\Gamma(1+(t+\nu)/2)\Gamma(1+(u+\nu)/2)} \quad (B.9a)$$

This expression may be rewritten in terms of the generalized hypergeometric functions yielding an interesting identity :

$$\begin{aligned} \frac{1}{|\nu| - t} {}_3F_2\left(\frac{s}{2}, \frac{s}{2} + |\nu|, \frac{|\nu| - t}{2}; 1\right) + \frac{1}{|\nu| + s + t} {}_3F_2\left(\frac{s}{2}, \frac{s}{2} + |\nu|, \frac{|\nu| + s + t}{2}; 1\right) \\ = \frac{\Gamma(1 - \frac{s}{2})\Gamma(|\nu| + 1)\Gamma(-\frac{t}{2} + \frac{|\nu|}{2})\Gamma(\frac{s+t+|\nu|}{2})}{\Gamma(|\nu| + \frac{s}{2})\Gamma(1 + \frac{|\nu| + t}{2})\Gamma(1 + \frac{|\nu| - s - t}{2})} \end{aligned} \quad (B.9b)$$

Analytic Continuation of the Mellin Transform

We shall now discuss the analytic structure of $f_{n\nu}(s, \alpha)$ and show that f extends to a meromorphic function throughout the complex plane with simple poles in α and s at all positive integers. We shall derive also the residues at these poles. We begin by splitting $f = f^+ + f^-$, with $0 < \delta < 1$:

$$\begin{aligned} f_{n\nu}^+(s, \alpha) &= C_{|\nu|}(s) \int_{\delta}^1 dx \ x^{-\alpha+n+|\nu|-1} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) \\ f_{n\nu}^-(s, \alpha) &= C_{|\nu|}(s) \int_0^{\delta} dx \ x^{-\alpha+n+|\nu|-1} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) \end{aligned} \quad (B.10)$$

Clearly, f^+ is an *entire* function of α , whereas f^- is an *entire* function of s . Analytic continuation of f^- in α is standard, and proceeds along the lines of Lemma I of §A. In this way we obtain an analytic continuation of f^- throughout the half-plane $\text{Re}(\alpha) < N$

$$\begin{aligned} f_{n\nu}^-(s, \alpha) &= \frac{\Gamma(\frac{\alpha-n-|\nu|}{2} - N + 1)}{\Gamma(\frac{\alpha-n-|\nu|}{2} + 1)} f_{n\nu}^{-(N)}(s, \alpha - 2N) \\ &\quad - \sum_{k=0}^{N-1} \frac{\Gamma(\frac{\alpha-n-|\nu|}{2} - k)}{\Gamma(\frac{\alpha-n-|\nu|}{2} + 1)} \delta^{-2\alpha+2k+2n+2|\nu|} F^{(k)}\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; \delta^2\right) \end{aligned} \quad (B.11)$$

Analytic continuation of $f_{n\nu}^+(s, \alpha)$ in s may be achieved by using the reflection formula (A.2), and we obtain

$$\begin{aligned} f_{n\nu}^+(s, \alpha) &= \frac{1}{2 \cos \frac{\pi s}{2}} \sum_{k=0}^{\infty} \left[C_k(s) C_{k+s-1}(2|\nu| + 2 - s) \int_{\delta}^1 dx \ x^{-\alpha-1+n+|\nu|} (1 - x^2)^k \right. \\ &\quad \left. - C_k(2|\nu| + 2 - s) C_{k-s+1}(s) \int_{\delta}^1 dx \ x^{-\alpha-1+n+|\nu|} (1 - x^2)^{k+1-s} \right] \end{aligned} \quad (B.12)$$

This series is absolutely convergent, since the integrals produce an exponential suppression factor of order $(1 - \delta^2)^k$ for large k ; of course, coefficients for small values of k may have poles themselves, and these must be isolated first.

Pole Structure and Residues

It is easy to see that poles in s occur only at positive integers. The structure of the residue depends upon whether s is even or odd. The poles at positive even integers s are similar in nature to the poles in α of the full function $f_{n\nu}(s, \alpha)$ and contribute to the poles in s in (B.9). The poles in s at positive odd integers have no analogues in terms of poles in α , and arise because the region around the singularity at $z = 1$ is integrated over only partially. Such poles cancel out of (B.9) between the two f terms.

The residues are defined as follows

$$F_\nu(N, \alpha) = \lim_{s \rightarrow N} (s - N) f_{0\nu}(s, \alpha) \quad (B.13)$$

and are given by the following expressions

$$\begin{aligned} F_\nu(2N, \alpha) &= -\frac{1}{2} \sum_{k=0}^{N-1} \frac{\Gamma(N - |\nu|) \Gamma(\frac{|\nu| - \alpha}{2})}{\Gamma(N) \Gamma(N - k - |\nu|) \Gamma(N - k) \Gamma(k + 1) \Gamma(\frac{|\nu| - \alpha}{2} + 2 + k - 2N)} \\ F_\nu(2N + 1, \alpha) &= \frac{1}{2\pi^2} \sum_{k=0}^{2N-1} \frac{\Gamma(N + |\nu| - \frac{1}{2} - k) \Gamma(\frac{|\nu| - \alpha}{2}) \Gamma(N - \frac{1}{2} - k) \Gamma(\frac{1}{2} - N)}{\Gamma(2N - k) \Gamma(-N + \frac{1}{2} + |\nu|) \Gamma(\frac{|\nu| - \alpha}{2} - k)} \end{aligned} \quad (B.14)$$

Using the binomial resummation formula

$$\frac{\Gamma(A + B - 1)}{\Gamma(A + B - N) \Gamma(N)} = \sum_{k=0}^{N-1} \frac{\Gamma(A) \Gamma(B)}{\Gamma(A - k) \Gamma(k + 1) \Gamma(B - N + 1 + k) \Gamma(N - k)} \quad (B.15)$$

we may rewrite the residue expression at even integers as follows, in terms of a factorized expression

$$\begin{aligned} F_\nu(2N, \alpha) &= - \frac{\Gamma(\frac{|\nu| - \alpha}{2}) \Gamma(\frac{-|\nu| - \alpha}{2})}{2\Gamma(N)^2 \Gamma(\frac{|\nu| - \alpha}{2} + 1 - N) \Gamma(\frac{-|\nu| - \alpha}{2} + 1 - N)} \\ &= - \frac{1}{2\Gamma(N)^2} \prod_{k=1}^{N-1} \left\{ k + \frac{\alpha + |\nu|}{2} \right\} \left\{ k + \frac{\alpha - |\nu|}{2} \right\} \end{aligned} \quad (B.16)$$

It does not seem that a similar resummation is possible for the residues at odd positive integers. Of course, as pointed out previously, the rôle of these poles is very different in string theory and we shall not make use of their expressions.

Inverse Laplace Transform

The inverse Laplace transform of (B.4) is defined as follows

$$e^{(n+|\nu|)\omega} C_{|\nu|}(s) F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; e^{-2\omega}\right) \theta(\omega) = \int_0^\infty d\beta e^{-\omega\beta} \varphi_{n\nu}(s, \beta) . \quad (B.17)$$

Formally, we have

$$\varphi_{n\nu}(s, \beta) = \sum_{k=0}^{\infty} C_k(s) C_{k+|\nu|}(s) \delta(2k + n + |\nu| - \beta) . \quad (B.18)$$

and from this expression, it is clear that one may obtain φ directly as a discontinuity of the Mellin transform:

$$\varphi_{n\nu}(s, \beta) \theta(\beta) = \frac{1}{2\pi i} \left(f_{n\nu}(s, \beta + i\epsilon) - f_{n\nu}(s, \beta - i\epsilon) \right) \quad (B.19)$$

This expression implies that $\varphi_{n\nu}(s, \beta)$ is entire as a function of s , since the poles of $f_{n\nu}$ cancel out. More precisely, since $\varphi_{n\nu}(s, \beta)$ should be interpreted as a distribution in β , the integrals of $\varphi_{n\nu}$ over any finite β interval are entire in s . However, integrals over $[0, \infty)$ produce poles in s at positive integers. We can see this explicitly in the following formulas which we require in Section III.2 and which can be obtained by differentiating (B.17) with respect to ω at $\omega = 0$

$$\begin{aligned} \int_0^\infty d\beta \varphi_{n\nu}(s, \beta) &= C_{|\nu|}(s) F\left(\frac{s}{2}, \frac{s}{2} + |\nu|, |\nu| + 1, 1\right) \\ \int_0^\infty d\beta \beta \varphi_{n\nu}(s, \beta) &= C_{|\nu|}(s) \left((n + |\nu|) F\left(\frac{s}{2}, \frac{s}{2} + |\nu|, |\nu| + 1, 1\right) \right. \\ &\quad \left. + F'\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1, 1\right) \right) \end{aligned} \quad (B.20)$$

In Section III.2, we actually also need to establish the meromorphicity in s of expressions of the form (B.18), with however additional integrations in the ω variable. The simplest such expression involves a single integration in ω , and can be rewritten as

$$\int_0^1 d\omega \int_0^\infty d\beta \varphi_{n\nu}(s, \beta) e^{-\omega\beta} = \int_0^1 d\omega \int_0^{2\pi} \frac{d\alpha}{2\pi} |1 - e^{i\alpha - \omega}|^{-s} e^{i\nu\alpha} e^{-n\omega} \quad (B.21)$$

in view of (B.4). In terms of the complex variable $w = \exp(i\alpha - \omega)$, the integral (B.21) reduces to an integral of the form (i) in Lemma A.3. The only difference is the change of domain of integration from $e^{-1} \leq |w| \leq 1$ to $1/2 \leq |w| \leq 1$, which gives rise only to a manifestly entire correction term. Thus (B.21) is again a meromorphic function of s , with poles at most at positive integers. A more complicated integral which may arise is one of the form

$$\int_0^1 \frac{d\omega}{\omega} \int_0^\infty d\beta \varphi_{n\nu}(s, \beta) (e^{-\omega\beta} - 1) = \int_0^1 dt \int_0^1 d\omega \int_0^\infty \varphi_{n\nu}(s, \beta) e^{-t\omega\beta} \quad (B.22)$$

The identity (B.4) leads now to integrals with the following type of singularities

$$\int_0^1 dt \int_0^1 d\omega (t^2 \omega^2 + \alpha^2 U^2)^{-s} E(t, \omega, \alpha) \quad (B.23)$$

where E, U are smooth functions of all variables, and U is non-vanishing. The scaling methods of Appendix A apply (c.f. Lemma A.4), and show that (B.23) is a meromorphic function of s . Evidently the above arguments will also establish the meromorphicity in s of any moments of $\varphi_{n\nu}$ of the form

$$\int_0^1 \frac{d\omega}{\omega^k} \int_0^\infty d\beta \varphi_{n\nu}(s, \beta) (e^{-\omega\beta} - \sum_{m=0}^{k-1} \frac{(-1)^m (\omega\beta)^m}{m!}) \quad (B.24)$$

since higher order Taylor expansions simply lead to the insertion of powers of $(1-t)$ in (B.23). Finally, the Mellin transform f is given by a Hilbert transform of the inverse Laplace transform

$$\int_0^\infty \frac{\varphi_{n\nu}(s; \tau - x)}{\tau - t} = f_{n\nu}(s, t - x) \quad (B.25)$$

This formula clearly exhibits the poles of $f_{n\nu}$ as a function of t .

Appendix C. EXPANSION POLYNOMIAL COEFFICIENTS

We need to determine three types of expansion coefficients for truncated ϑ functions both for the Type II and for the heterotic strings.

Type II Superstrings

First the monomial expansion

$$\prod_{i=1}^4 \int_0^{2\pi} \frac{d\alpha_i}{2\pi} \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \sum_{n_i=0}^\infty P_{n_i}^{(0)}(s, t) |q|^{\sum_i n_i u_i} \quad (C.1)$$

Second the expansion with two factors isolated

$$\begin{aligned} \int_0^{2\pi} \frac{d\alpha_1}{2\pi} \int_0^{2\pi} \frac{d\alpha_3}{2\pi} \mathcal{R}(|q|^{u_i}, \alpha_i; s, t) &= \prod_{i=2,4} |1 - e^{i\alpha_i} |q|^{u_i}|^{-s} \times \sum_{n_i=0}^\infty \sum_{|\nu_i| \leq n_i} P_{n_i; \nu_2, \nu_4}^{(2)}(s, t) \\ &\quad |q|^{\sum_i n_i u_i} e^{i\nu_2 \alpha_2 + i\nu_4 \alpha_4} \end{aligned} \quad (C.2)$$

Third, the expansion coefficients with four factors isolated

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \prod_{i=1}^4 |1 - e^{i\alpha_i} |q|^{u_i}|^{-s_i} \sum_{n_i=0}^\infty \sum_{|\nu_i| \leq n_i} P_{n_i; \nu_i}^{(4)}(s, t) |q|^{\sum_i n_i u_i} e^{i \sum_i \nu_i \alpha_i} \quad (C.3)$$

To do so, we recast the function \mathcal{R} of (6.5) in terms of a product of simple functions

$$T(z, q; s) = \prod_{n=0}^{\infty} (1 - q^n z)^{-s/2} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{k,l}(s) q^k z^l \quad (C.4)$$

$$T^*(z, q; s) = \prod_{n=1}^{\infty} (1 - q^n z)^{-s/2} = \sum_{k=0}^{\infty} \sum_{l=0}^k T_{k,l}^*(s) q^k z^l$$

We recall that the modulus q is not an independent variable, but is given by (3.12). As a result, we have the simple expression

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \prod_{i=1}^{12} |T(w_i, q; s_i)|^2 \quad (C.5)$$

Here, we have defined w_i as in (3.6) and (3.13) s_i as in (3.15). Upon using the expansion of (C.4) in (C.1-3), we shall have to collect powers of w_i , which are given by the following sums

$$\begin{aligned} L_1 &= l_1 + l_5 + l_8 + l_9 + l_{10} + l_{12} + \sum_i k_i \\ L_2 &= l_2 + l_5 + l_6 + l_7 + l_9 + l_{10} + \sum_i k_i \\ L_3 &= l_3 + l_6 + l_7 + l_8 + l_{10} + l_{11} + \sum_i k_i \\ L_4 &= l_4 + l_7 + l_8 + l_9 + l_{11} + l_{12} + \sum_i k_i \end{aligned} \quad (C.6)$$

and powers \bar{L}_i of \bar{w}_i are given by barred quantities in sums (C.6).

We may now easily write down the expansions of (C.1-4) in terms of the expansion coefficients T and T^* . First, in terms of monomials only

$$P_{n_i}^{(0)}(s, t) = \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \prod_{j=1}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j) \quad (C.7a)$$

with the following restrictions on the sums

$$n_i = 2L_i = 2\bar{L}_i \quad i = 1, 2, 3, 4 \quad (C.7b)$$

Second, the expansion retaining two factors is given by

$$P_{n_i; \nu_2 \nu_4}^{(2)}(s, t) = \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \prod_{\alpha=2,4} T_{k_\alpha l_\alpha}^*(s) T_{\bar{k}_\alpha \bar{l}_\alpha}^*(s) \prod_{j \neq 2,4}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j) \quad (C.8a)$$

with the following restrictions on the sums

$$\begin{aligned} i=1,3 & \quad n_i = 2L_i = 2\bar{L}_i \\ i=2,4 & \quad n_i + \nu_i = 2L_i \quad n_i - \nu_i = 2\bar{L}_i \end{aligned} \quad (C.8b)$$

Third, the expansion coefficients of the expansion with four factors isolated are given by

$$P_{n_i; \nu_i}^{(4)}(s, t) = \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \prod_{\alpha=1}^4 T_{k_\alpha l_\alpha}^*(s_\alpha) T_{\bar{k}_\alpha \bar{l}_\alpha}^*(s_\alpha) \prod_{j=5}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j) \quad (C.9a)$$

with the restrictions on the sums

$$n_i + \nu_i = 2L_i \quad n_i - \nu_i = 2\bar{L}_i \quad i = 1, 2, 3, 4 \quad (C.9b)$$

Heterotic Superstrings

We shall furthermore provide analogous formulas for the expansion coefficients that occur in the heterotic four point amplitudes for the scattering of gauge bosons. These coefficients are defined by the following expressions First the monomial expansion

$$\prod_{i=1}^4 \int_0^{2\pi} \frac{d\alpha_i}{2\pi} \mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T) = \sum_{n_i=0}^{\infty} P_{n_i}^{(0)H}(s, t; S, T) |q|^{\sum_i n_i u_i} \quad (C.10)$$

Second the expansion with two factors isolated

$$\begin{aligned} \int_0^{2\pi} \frac{d\alpha_1}{2\pi} \int_0^{2\pi} \frac{d\alpha_3}{2\pi} \mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T) \\ = \prod_{i=2,4} (1 - e^{i\alpha_i} |q|^{u_i})^{-s/2} (1 - e^{-i\alpha_i} |q|^{u_i})^{-s/2 - S/2 - 2} \\ \times \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{n_i; \nu_2, \nu_4}^{(2)H}(s, t; S, T) |q|^{\sum_i n_i u_i} e^{i\nu_2 \alpha_2 + i\nu_4 \alpha_4} \end{aligned} \quad (C.11)$$

Third, the expansion coefficients with four factors isolated

$$\begin{aligned} \mathcal{R}^H(|q|^{u_i}, \alpha_i; s, t; S, T) = \prod_{i=1}^4 (1 - e^{i\alpha_i} |q|^{u_i})^{-s_i/2} (1 - e^{-i\alpha_i} |q|^{u_i})^{-s_i/2 - S_i/2 - 2} \\ \times \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{n_i; \nu_i}^{(4)H}(s, t) |q|^{\sum_i n_i u_i} e^{i \sum_i \nu_i \alpha_i} \end{aligned} \quad (C.12)$$

We may now write down the expansions of (C.8-11) in terms of the expansion coefficients T and T^* . First for the monomial terms only

$$P_{n_i}^{(0)H}(s, t; S, T) = \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \sum_{p \in \Lambda} \sum_{n=0}^{\infty} f_n \prod_{j=1}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j + S_j + 4) \quad (C.13a)$$

with the following restrictions on the sums

$$\begin{aligned}
n_1 &= 2L_1 = 2\bar{L}_1 + p^2 + 2n - 2 \\
n_2 &= 2L_2 = 2\bar{L}_2 + (p - K_1)^2 + 2n - 2 \\
n_3 &= 2L_3 = 2\bar{L}_3 + (p + K_3 + K_4)^2 + 2n - 2 \\
n_4 &= 2L_4 = 2\bar{L}_4 + (p + K_4)^2 + 2n - 2
\end{aligned} \tag{C.13b}$$

Second, the expansion retaining two factors is given by

$$\begin{aligned}
P_{n_i; \nu_2 \nu_4}^{(2)H}(s, t; S, T) &= \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \sum_{p \in \Lambda} \sum_{n=0}^{\infty} f_n \prod_{\alpha=2,4} T_{k_\alpha l_\alpha}^*(s) T_{\bar{k}_\alpha \bar{l}_\alpha}^*(s + S + 4) \\
&\quad \prod_{j \neq 2,4}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j + S_j + 4)
\end{aligned} \tag{C.14a}$$

with the following restrictions on the sums

$$\begin{aligned}
n_1 &= 2L_1 = 2\bar{L}_1 + p^2 + 2n - 2 \\
n_3 &= 2L_3 = 2\bar{L}_3 + (p + K_3 + K_4)^2 + 2n - 2 \\
n_2 + \nu_2 &= 2L_2 \quad n_2 - \nu_2 = 2\bar{L}_2 + (p - K_1)^2 + 2n - 2 \\
n_4 + \nu_4 &= 2L_4 \quad n_4 - \nu_4 = 2\bar{L}_4 + (p + K_4)^2 + 2n - 2
\end{aligned} \tag{C.14b}$$

Third, the expansion coefficients of the expansion with four factors isolated are given by

$$\begin{aligned}
P_{n_i; \nu_i}^{(4)H}(s, t; S, T) &= \sum_{k_i, \bar{k}_i \geq 0} \sum_{l_i, \bar{l}_i \geq 0} \sum_{p \in \Lambda} \sum_{n=0}^{\infty} f_n \prod_{\alpha=1}^4 T_{k_\alpha l_\alpha}^*(s_\alpha) T_{\bar{k}_\alpha \bar{l}_\alpha}^*(s_\alpha + S_\alpha + 4) \\
&\quad \prod_{j=5}^{12} T_{k_j l_j}(s_j) T_{\bar{k}_j \bar{l}_j}(s_j + S_j + 4)
\end{aligned} \tag{C.15a}$$

with the restrictions on the sums

$$\begin{aligned}
n_1 - \nu_1 &= 2\bar{L}_1 + p^2 + 2n - 2 \\
n_2 - \nu_2 &= 2\bar{L}_2 + (p - K_1)^2 + 2n - 2 \\
n_3 - \nu_3 &= 2\bar{L}_3 + (p + K_3 + K_4)^2 + 2n - 2 \\
n_4 - \nu_4 &= 2\bar{L}_4 + (p + K_4)^2 + 2n - 2 \\
n_i + \nu_i &= 2L_i \quad i = 1, 2, 3, 4
\end{aligned} \tag{C.15b}$$

*Recursion Relations for Expansion Coefficients T and T^**

It remains to determine the coefficient functions $T_{kl}(s)$ and $T_{kl}^*(s)$. This may simply be achieved by obtaining a functional relation

$$\begin{aligned} T(qz, q; s) &= (1 - z)^{s/2} T(z, q; s) \\ T^*(qz, q; s) &= (1 - zq)^{s/2} T^*(z, q; s) \end{aligned} \quad (C.16)$$

These relations translate into recursion relations on the coefficients T and T^* as follows:

$$T_{k,l}(s) = \sum_{p=0}^l C_{l-p}(s) T_{k-p,p}(s) \quad (C.17)$$

$$T_{k,l}^*(s) = \sum_{p=0}^{k-l} C_{l-p}(s) T_{k-l,p}^*(s)$$

supplemented with the following boundary conditions

$$T_{k,0}(s) = \delta_{k,0} \quad T_{0,0}^*(s) = 1 \quad (C.18a)$$

Furthermore, we should note that by construction, we have

$$\begin{aligned} T_{k,l} &= 0 \quad T_{k,l}^* = 0 \quad \text{for } k < 0 \text{ or } l < 0 \\ T_{k,l} &= 0 \quad \text{for } k < l \end{aligned} \quad (C.18b)$$

It is easy to produce explicit solutions for the first few values of l and all values of k , say for the coefficients T . It is convenient to express these in terms of the binary function

$$\theta_k(n) = \begin{cases} 1 & \text{if } k \text{ divides } n \\ 0 & \text{if } k \text{ does not divide } n \end{cases} \quad (C.19)$$

We find

$$\begin{aligned} T_{k,1}(s) &= C_1(s) \\ T_{k,2}(s) &= \theta_2(k) C_2(s) + \frac{1}{2}(k+1 - \theta_2(k)) C_1(s)^2 \\ T_{k,3}(s) &= \theta_3(k) C_3(s) + \frac{1}{2}(3\theta_3(k-2) - \theta_2(k-1) + 2\theta_3(k-1) + k) C_1(s) C_2(s) \\ &\quad + \frac{1}{12}(k^2 - 1 - 3\theta_2(k) + 4\theta_3(k)) C_1(s)^3 \end{aligned} \quad (C.20)$$

Appendix D: ANALYTIC STRUCTURE OF THE BOX DIAGRAM

We consider ϕ^3 scalar field theory in d space-time dimensions, and evaluate the box diagram with arbitrary mass parameters. The four external particles are taken to be on-shell and assumed to be massless. It is easy to write the amplitude using Feynman parameters with the conventions of Fig. 5.

$$T(s, t) = 3! \int \frac{d^d k}{(2\pi)^d} \int_0^1 du_i \frac{\delta(1 - u_1 - u_2 - u_3 - u_4)}{[k^2 + \sum_i m_i^2 u_i - u_1 u_3 s - u_2 u_4 t]^4} \quad (D.1)$$

With the help of the exponential representation of the denominator, and upon carrying out the k -integration, we get

$$T(s, t) = \frac{(2\pi)^4}{(8\pi^2)^{d/2}} \int_0^\infty \frac{d\tau}{\tau^{d/2-3}} \int_0^1 du_i \delta(1 - \sum_i u_i) \exp\{-2\pi\tau[\sum_i u_i m_i^2 - u_1 u_3 s - u_2 u_4 t]\} \quad (D.2)$$

Remarkably, this amplitude has the same structure as the string amplitude of (4.4), but with fixed value of $\beta_i = m_i^2$ and full range for τ from 0 to ∞ and space-time dimension $d = 10$. Despite these differences, we may use the steps explained in §IV, (4.7) – (4.17) to derive the double spectral density of the box diagram. We find

$$T(s, t) = \frac{(4\pi)^{-d/2}}{\Gamma(\frac{d}{2} - 2)} \int_0^1 du_1 \int_0^{1-u_1} du_2 (1 - u_1 - u_2)^{-3+d/2} \int_0^\infty dx x^{\frac{d}{2}-3} \times [(x + x_0 + m_3^2 - su_1)(x + x_0 + m_4^2 - tu_2)]^{-1} \quad (D.3a)$$

with

$$x_0 = \frac{u_1 m_1^2 + u_2 m_2^2}{1 - u_1 - u_2} \quad (D.3b)$$

The four point amplitude may be represented by a double dispersion relation in the following way

$$T(s, t) = \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho_{\{m_i^2\}}(\sigma, \tau)}{(\sigma - s)(\tau - t)} \quad (D.4)$$

and the double spectral density is given by

$$\rho_{\{m_i^2\}}(\sigma, \tau) = \frac{(4\pi)^{2-d/2}}{4\Gamma(\frac{d}{2} - 2)} \int_0^1 du_1 \int_0^1 du_2 \theta(1 - u_1 - u_2) (1 - u_1 - u_2)^{-3+d/2} \times \int_{x_0}^\infty dx (x - x_0)^{-3+\frac{d}{2}} \delta(x + m_3^2 - \sigma u_1) \delta(x + m_4^2 - \tau u_2) \quad (D.5)$$

It is easy to carry out the u_1 and u_2 integrations:

$$\rho_{\{m_i^2\}}(\sigma, \tau) = \frac{(4\pi)^{2-d/2}}{4\Gamma(\frac{d}{2} - 2) \sigma \tau} \int_0^\infty dx \theta\left(1 - \frac{x + m_3^2}{\sigma} - \frac{x + m_4^2}{\tau}\right) \quad (D.6)$$

where M is a quadratic function of x :

$$M \equiv -\frac{\sigma + \tau}{\sigma\tau}(x^2 - 2xA + B^2) \quad (D.7a)$$

with

$$A = \frac{\sigma\tau - (m_1^2 + m_3^2)\tau - (m_2^2 + m_4^2)\sigma}{2(\sigma + \tau)} \quad (D.7b)$$

$$B^2 = \frac{m_1^2 m_3^2 \tau + m_2^2 m_4^2 \sigma}{\sigma + \tau} \quad (D.7c)$$

It is clear that $M > 0$ implies that the θ -function in (D.6) equals 1 as long as $x, \sigma, \tau > 0$. Hence the first θ function under the integral is redundant. Now the x -integral is easily performed, and we find

$$\rho_{\{m_i^2\}}(\sigma, \tau) = \frac{4\pi^{5/2}}{(4\pi)^{d/2}\Gamma(\frac{d-3}{2})} \frac{(\sigma + \tau)^{\frac{d}{2}-3}}{(\sigma\tau)^{\frac{d}{2}-2}} (A^2 - B^2)^{\frac{d-5}{2}} \theta(A^2 - B^2) \theta(A) \quad (D.8)$$

The spectral representation (D.4) strictly speaking only holds for $d < 8$, at which point ultraviolet divergences occur and subtractions are required. In dimension d , there are $[\frac{d-6}{2}]$ subtractions necessary, which may be handled by dimensional regularization. For the case $d = 4$, this result coincides with the one given by Itzykson and Zuber [21] for massless external particles.

Support of the Spectral Density for the Box Diagram.

The support of the spectral density $\rho_{m_i^2}$ is determined by the inequalities resulting from the θ functions in (D.8):

$$A \geq 0 \quad A^2 - B^2 \geq 0 \quad \sigma, \tau \geq 0 \quad (D.9)$$

This support region is the same in all dimensions of space-time $d \geq 4$, and a non-zero ρ_{st} results in a branch cut in the s, t complex planes. We shall now solve the above inequalities and parametrize the domain explicitly. We begin by solving $A^2 - B^2 \geq 0$, temporality ignoring the remaining 3 constraints. Though the inequality appears to be quartic in σ and τ , it is in fact quadratic in $1/\sigma$ and $1/\tau$. Introducing an arbitrary mass scale m^2 , we can recast $A^2 - B^2 \geq 0$ in the form

$$\begin{aligned} \left(1 - \frac{m_1^2 + m_3^2}{\sigma} - \frac{m_2^2 + m_4^2}{\tau}\right)^2 + \left(\frac{m_1^2 m_3^2 - m^4}{\sigma m^4} + \frac{m_2^2 m_4^2 - m^2}{\tau m^2}\right)^2 \\ \geq \left(\frac{m_1^2 m_3^2 + m^4}{\sigma m^2} + \frac{m_2^2 m_4^2 + m^4}{\tau m^2}\right)^2 \end{aligned} \quad (D.10)$$

This region can be parametrized by two real numbers x, y , satisfying $x^2 + y^2 \geq 1$:

$$\begin{aligned} \frac{m_1^2 m_3^2 - m^4}{\sigma} + \frac{m_2^2 m_4^2 - m^4}{\tau} &= x \left(\frac{m_1^2 m_3^2 + m^4}{\sigma} + \frac{m_2^2 m_4^2 + m^4}{\tau} \right) \\ 1 - \frac{m_1^2 + m_3^2}{\sigma} - \frac{m_2^2 + m_4^2}{\tau} &= y \left(\frac{m_1^2 m_3^2 + m^4}{\sigma m^2} + \frac{m_2^2 m_4^2 + m^4}{\tau m^2} \right) \end{aligned} \quad (D.11)$$

The additional requirements that $\sigma, \tau \geq 0$ imply the obvious restriction that $x_- \leq x \leq x_+$ with

$$x_{\pm} = \max_{\min} \left[\frac{m_1^2 m_3^2 - m^4}{m_1^2 m_3^2 + m^4}, \frac{m_2^2 m_4^2 - m^4}{m_2^2 m_4^2 + m^4} \right] \quad (D.12)$$

The restriction that $A \geq 0$ implies $y \geq 0$ in view of $\sigma, \tau \geq 0$. It is now straightforward to solve for σ and τ and we find

$$\begin{cases} \sigma = +\lambda A_2^{-1} \\ \tau = -\lambda A_1^{-1} \end{cases} \quad \begin{cases} A_1 = m_1^2 m_3^2 - m^4 - x(m_1^2 m_3^2 + m^4) \\ A_2 = m_2^2 m_4^2 - m^4 - x(m_2^2 m_4^2 + m^4) \\ \lambda = (m_1^2 + m_3^2)A_2 - (m_2^2 + m_4^2)A_1 - 2yDm^2 \\ D = m_1^2 m_3^2 - m_2^2 m_4^2 \end{cases} \quad (D.13)$$

The scale m^2 is arbitrary in this parametrization and may be picked in the most convenient way; for example if $m_1^2 m_3^2 > m_2^2 m_4^2$, we may choose $m^4 = m_2^2 m_4^2$, so that $x_- = 0$. Notice that when $D > 0$, we have $A_1 \geq 0, A_2 \leq 0$ and hence $\lambda \leq 0$; on the other hand when $D < 0$, then $A_1 \leq 0, A_2 \geq 0$ and hence $\lambda \geq 0$.

When $D = 0$, the above parametrization is degenerate and this case may be treated separately. The function B^2 is independent of σ and τ and we get

$$\rho_{\{m_i^2\}}(\sigma, \tau) = \frac{2^{7-d} \pi^{5/2}}{(4\pi)^{d/2} \Gamma(\frac{d-3}{2})} \frac{(\sigma\tau)^{\frac{d}{2}-3}}{(\sigma + \tau)^{\frac{d}{2}-2}} \prod_{\epsilon=\pm} \left(1 - \frac{(m_1 + \epsilon m_3)^2}{\sigma} - \frac{(m_2 + \epsilon m_4)^2}{\tau} \right)^{\frac{d-5}{2}} \quad (D.14)$$

and the region of (D.9) corresponds to a hyperbola in σ, τ space

$$1 - \frac{(m_1 + m_3)^2}{\sigma} - \frac{(m_2 + m_4)^2}{\tau} \geq 0 \quad (D.15)$$

which is represented in Fig. 7. This region is easily parametrized by standard methods, which we shall not elaborate on here. It is easy to see that all these results on the box diagram agree with those given in [21].

Parametrization of the Box Graph Amplitude.

The amplitude for the box graph may now be recovered by using the double spectral integral (D.4). For the case $D \neq 0$, the parametrization of (D.13) yields a natural solution to the θ -function inequalities of (D.8), and we get

$$T(s, t) = 4m^{2d-4} |D|^{-1+\frac{d}{2}} \int_{x_-}^{x_+} dx \int_0^\infty dy \frac{\theta(x^2 + y^2 - 1)(x^2 + y^2 - 1)^{\frac{d-5}{2}}}{|\lambda|^{\frac{d}{2}-2} (\lambda - sA_2)(\lambda + tA_1)} (1-x)^{2-\frac{d}{2}} \quad (D.16)$$

The x and y integrations may be decoupled by introducing the variable $y = (1 - x^2)^{\frac{1}{2}} z$, and making the definitions:

$$\begin{aligned} c &= \frac{(m_2^2 + m_4^2)A_1 - (m_1^2 + m_3^2)A_2}{2D(1-x^2)^{\frac{1}{2}}} \geq 0 \\ a &= c + \frac{sA_2}{2D(1-x^2)^{\frac{1}{2}}} \quad b = c - \frac{tA_1}{2D(1-x^2)^{\frac{1}{2}}} \end{aligned} \quad (D.17)$$

Assuming that the dimension of space-time is even $d/2 - 2 = k$, and k integer $k \geq 0$, we may rewrite T in the following form:

$$T(s, t) = 2^{-d/2} m^d |D| \int_{x_-}^{x_+} dx (1-x)^{-\frac{1}{2}-\frac{d}{4}} (1+x)^{-\frac{5}{2}+\frac{d}{4}} I_k(a, b, c) \quad (D.18)$$

where the integral I_k is defined as

$$I_k(a, b, c) \equiv \int_1^\infty dz \frac{(z^2 - 1)^{k-\frac{1}{2}}}{(z+a)(z+b)(z+c)^k} \quad (D.19)$$

The latter integral is elementary, and we have

$$I_k(a, b, c) = \frac{1}{(k-1)!} \left(-\frac{\partial}{\partial c} \right)^{k-1} \left\{ \frac{J_k(a) - J_k(c)}{(a-b)(a-c)} + \frac{J_k(b) - J_k(c)}{(b-a)(b-c)} \right\} \quad (D.20)$$

with $J_k(a) = 2(a^2 - 1)^{k-\frac{1}{2}} \text{Argch}(a)$. We have omitted in $I_k(a, b, c)$ large z subtractions that render the integral convergent. There are k subtractions to be made which are polynomial in a, b and c , which means polynomial in s, t and u , and hence correspond to the usual ultraviolet subtractions in dispersion relations.

For the case $D = 0$, this answer may be considerably simplified, and may be obtained by taking the limit $D \rightarrow 0^+$ in (D.16). It is convenient to set $m_1 m_3 = m^2$, so that $x_- = 0$ and $x_+ \sim D \rightarrow 0$. We find

$$T(s, t) = 2(m_1 m_3)^{d-4} \int_0^1 d\alpha \int_1^\infty dy \frac{(y^2 - 1)^{\frac{d-5}{2}}}{|\mu|^{\frac{d}{2}-2} (\mu - \alpha s)(\mu - (1-\alpha)t)} \quad (D.21)$$

where we have used the abbreviation

$$\mu = (m_1^2 + m_3^2)\alpha + (m_2^2 + m_4^2)(1-\alpha) + 2ym_1 m_3 \quad (D.22)$$

This integral is easily evaluated using the formulas (D.19) and (D.20).

Appendix E: FORWARD-LIKE SCATTERING AMPLITUDES

The forward scattering amplitude is given by $A_{\{n_i \nu_i\}}(s, t)$ with $t = 0$. Its analytic continuation is considerably simpler than that of the full amplitude $A_{\{n_i \nu_i\}}(s, t)$. In this Appendix we show how to analytically continue forward-like scattering amplitudes, and how to actually obtain $A_{\{n_i \nu_i\}}(s, t)$ from them. In effect, this provides another method of analytic continuation, which is more general than that of dispersion relations and is perhaps easier to generalize to the case of higher point functions. For simplicity, we restrict ourselves to the case $n_i = \nu_i = 0$, and denote the amplitude $A_{\{n_i \nu_i\}}(s, t)$ by $A(s, t)$.

The forward-like scattering amplitudes $A_{kl}(s; S, T_0)$ we need are defined as follows

$$A_{kl}(s; S, T_0) = \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) |q|^{-(Su_1u_3 + T_0u_2u_4 + ku_2 + lu_4)} \prod_{m=0}^1 F\left(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_{2m}}\right) \quad (E.1)$$

for some fixed value of the parameter T_0 between 0 and -1, and k, l positive integers. We also require

$$A(s, t; S, T) = \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int \prod_{i=1}^4 du_i \delta(1 - \sum_{i=1}^4 u_i) |q|^{-(Su_1u_3 + Tu_2u_4)} \times \prod_{i=1}^4 F\left(\frac{s_i}{2}, \frac{s_i}{2}; 1; |q|^{2u_i}\right) \quad (E.2)$$

Evidently

$$A(s, t; S, T)|_{S=s, T=t} = A(s, t)$$

so our problem is to analytically continue $A(s, t; S, T)$ in all variables. For this we require the following properties of $A(s, t; S, T)$

- (i) $\frac{\partial^{k+l} A}{\partial S^k \partial T^l}(s, t; S, T)|_{S=S_0, T=T_0}$ is a globally meromorphic function of s, t for all k, l ;
- (ii) For each n , the derivatives $\frac{\partial^{2n} A}{\partial^n S \partial^n T}(s, t; S, T)$ can be analytically continued to the half-space $\text{Re } S, \text{Re } T < 2 + 3n/4$ cut along the positive real axis and contain no poles in this region;
- (iii) For any fixed S_0, T_0 between 0 and -1, and for s, t varying only in an arbitrary fixed half-space $\text{Re } s, \text{Re } t < 2(N+1)$, we have

$$A(s, t; S, T_0) = \sum_{k,l=0}^\infty C_k(s) C_l(t) A_{kl}(s; S, T_0) + M_N(s, t)$$

where $M_N(s, t)$ is a meromorphic function of both s and t in the half-space $\text{Re } s, \text{Re } t < 2(N+1)$. Similar relations hold between the derivatives $\frac{\partial^m A}{\partial T^m}(s, t; S, T)|_{T=T_0}$ and expressions of the form (C.1) with an additional insertion $(u_1u_3)^k$ in the integrand.

Assuming (i)-(iii) for the moment, we make use of the following version of Taylor's formula, applied to $A(s, t; S, T)$ as a function of S and T :

$$\begin{aligned} A(s, t; S, T) = & - \sum_{k,l=0}^{N-1} \frac{S^k T^l}{k!l!} \frac{\partial^{k+l} A}{\partial S^k \partial T^l}(s, t; S_0, T_0) \\ & + \sum_{k=0}^{N-1} \frac{S^k}{k!} \frac{\partial^k A}{\partial S^k}(s, t; S_0, T) + \sum_{k=0}^{N-1} \frac{T^k}{k!} \frac{\partial^k A}{\partial T^k}(s, t; S, T_0) \\ & + \left[\frac{1}{(n-1)!} \right]^2 \int_{S_0}^S dS_1 \int_{T_0}^T dT_1 (S - S_1)^{n-1} (T - T_1)^{n-1} \frac{\partial^{2n} A}{\partial S^n \partial T^n}(s, t; S_1, T_1) \end{aligned} \quad (E.3)$$

The first term on the right hand side of (E.3) is meromorphic by (i), while the last term admits a holomorphic continuation in the S and T planes cut along the real positive axis by (ii). Thus it suffices to analytically continue the third term in (E.3), the second term being similar with the roles of s, S and t, T interchanged. In view of (iii), this reduces to the analytic continuation of the forward-like scattering amplitudes $A_{kl}(s; S, T_0)$, as we had stated earlier.

We sketch now the arguments for (i)-(iii) and an analytic continuation for $A_{kl}(s, t)$.

Verification of (i)-(iii).

For S_0 and T_0 between -1 and 0 , the factor $|q|^{-S_0 u_2 u_4 - T_0 u_1 u_3}$ remains bounded. The du_i integrals of hypergeometric functions just produce poles, as we saw in Appendix B. This implies (i). For (ii) we need an intuitive understanding of how poles on top of cuts emerge. The cuts arise from the analytic continuation of the $d\tau_2$ integral. To perform this integral, we need to expand the hypergeometric functions in series in $|q|^{2u_i}$. But the series after the integral is performed converge only when $\text{Re } s, \text{Re } t < 2$. This is how the poles at $s = 2$ and $t = 2$ manifest themselves. To obtain a better radius of convergence, we consider instead the derivatives $\partial^{2n}/\partial S^n \partial T^n$. Each derivative $\partial^2/\partial S \partial T$ brings down a factor $\prod_{i=1}^4 u_i$, which extends simultaneously the radii of convergence of all the factors involved by $3/4$. Our general statement for any n follows at once.

To see (iii), we exploit the $u_1 \leftrightarrow u_3$ symmetry to restrict the region of integration to $0 \leq u_1 \leq u_3$. When $|q|^{2u_1} \geq e^{-2\pi}$, the integrand of $A(s, t; S, T_0)$ is bounded and hence the contribution of this region is a global meromorphic function. When $|q|^{2u_1} \leq e^{-2\pi}$, we also have $|q|^{2u_3} \leq e^{-2\pi}$. Both hypergeometric functions can now be expanded into series which are convergent for all t

$$\begin{aligned} \prod_{m=1}^2 F\left(\frac{t}{2}, \frac{t}{2}; 1; |q|^{2u_{2m+1}}\right) &= \sum_{k,l=0}^N C_k(t) C_l(t) |q|^{2ku_1 + 2lu_3} \\ &\quad + |q|^{2(N+1)u_1} e_N(|q|^{2u_1}) F\left(\frac{t}{2}, \frac{t}{2}; 1; |q|^{2u_3}\right) \\ &\quad + |q|^{2(N+1)u_3} e_N(|q|^{2u_3}) F\left(\frac{t}{2}, \frac{t}{2}; 1; |q|^{2u_1}\right) \\ &\quad + |q|^{2(N+1)(u_1+u_3)} e_N(|q|^{2u_1}) e_n(|q|^{2u_3}) \end{aligned}$$

The contributions of the last three terms in $A(s, t; S, T_0)$ are meromorphic in the half-plane $\text{Re } s, \text{Re } t < 2(N+1)$. As for the first term, the symmetry $u_1 \leftrightarrow u_3$ allows us to restore the region of integration back to the full region $0 \leq u_1, u_3 \leq 1$, up to a globally meromorphic function. We recognize then the contribution of each term to $A(s, t; S, T_0)$ to be the expression $A_{kl}(s, t; T_0)$ of (E.1).

Analytic Continuation of $A_{kl}(s; S, T_0)$.

Our strategy is to isolate within the integral expression for $A_{kl}(s; S, T_0)$ the region which produces double poles on top of cuts. For this we begin by rewriting $A_{kl}(s; S, T_0)$

as

$$A_{kl}(s; S, T_0) = \int_1^\infty \frac{d\tau_2^2}{\tau_2^2} \int_0^\infty du_1 du_2 \theta(1 - u_1 - u_2) |q|^{(-Su_1+2l)(1-u_1-u_2)+2ku_1} F\left(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_2}\right) \\ \times \int_0^1 \frac{d\lambda_4}{(\ln|q|)^2} \theta(\lambda_4 - |q|^{2(1-u_1-u_2)}) \lambda_4^{-1-\frac{1}{2}T_0u_2+\frac{1}{2}Su_1} F\left(\frac{s}{2}, \frac{s}{2}; 1; \lambda_4\right)$$

Next we break up the region of integration as follows:

$$\begin{aligned} \text{(I)} : & |q|^{2(1-u_1-u_2)} \leq \lambda_4 \leq 1/2 \\ \text{(II)} : & |q|^{2(1-u_1-u_2)} \leq 1/2 \leq \lambda_4 \\ \text{(III)} : & 1/2 \leq |q|^{2(1-u_1-u_2)} \end{aligned}$$

In the region (III), the exponential factor $|q|^{(-Su_1+2l)(1-u_1-u_2)+2ku_1}$ remains bounded. Thus this region contributes a meromorphic function, and we need only consider the regions (I) and (II).

We divide further (II) into

$$\begin{aligned} \text{(II.1)} : & 2|q|^{2(1-u_1)} \leq |q|^{2u_2} \leq 1/2 \\ \text{(II.2)} : & 2|q|^{2(1-u_1)} \leq 1/2 \leq |q|^{2u_2} \\ \text{(II.3)} : & 1/2 \leq |q|^{2(1-u_1)} \end{aligned}$$

Again the third region (II.3) contributes a meromorphic function. To treat (II.2), we introduce the new variable $\lambda_2 = |q|^{2u_2}$. If T_0 were 0, we would be able to carry out the du_2 integral explicitly and write (II.2) in terms of $f_-(s, -\frac{1}{2}Su_1 + l)f_-(s, -\frac{1}{2}Su_1 + l)|q|^{-su_1(1-u_1)}$. The integral in $|q|$ can then also be carried out explicitly, yielding cuts in terms of $\ln(-su_1(1-u_1))$ and hence $\ln(-s)$. When T_0 is not 0, the integral du_2 can no longer be carried out explicitly, but the $|q|$ integral needs to be done in order to exhibit the cut in s . To handle this technical difficulty, we have to

- i. Integrate *first* in $|q|$ to get the analytic cut, and *last* in λ_4 ;
- ii. Show that the resulting dependence on λ_2 and u_1 is C^∞ ;
- iii. From this C^∞ dependence in λ_2 and the $d\lambda_2$ integral, produce finally the double poles in s .

Thus we introduce $x \equiv 2\pi(1-u_1)\tau_2 = -\frac{1}{2}(1-u_1)\ln|q|^2$ and rewrite the contribution from (II.2) as

$$\begin{aligned} \text{(II.2)} = & \frac{\pi}{2} \int_0^1 du_1 (1-u_1)^3 \int_{1/2}^1 d\lambda_2 \lambda_2^{-1+\frac{1}{2}Su_1} F_s(\lambda_2) \int_{1/2}^1 d\lambda_4 \lambda_4^{-1+(\frac{1}{2}Su_1-l)} F_s(\lambda_4) \\ & \times \int_{\ln\sqrt{2}}^\infty \frac{dx}{x^4} \exp\left(-x(-Su_1 + 2l + 2k\frac{u_1}{1-u_1}) - \frac{1}{2x}(1-u_1)T_0(\ln\lambda_2)(\ln\lambda_4)\right) \end{aligned}$$

Now the integral in x converges initially for $Res \leq 0$ only. However, as we shall see in Appendix F, we can analytically continue it into

$$\Psi_4(-Su_1 + 2l + 2k \frac{u_1}{1-u_1}, T_0(1-u_1)\ln\lambda_2 \ln\lambda_4)$$

for $s \in \mathbf{C} \setminus \mathbf{R}_+$, $T_0 \in \mathbf{C}$, with Ψ_4 the function defined in Lemma F.1. Furthermore, for $T_0 \in \mathbf{R}_-$, the resulting expression is uniformly bounded in u_1 and smooth in $\ln\lambda_2, \ln\lambda_4$. Since we are in the range $1/2 \leq \lambda_2, \lambda_4 \leq 1$, this means that it is smooth in λ_2 and λ_4 . It follows from Lemma 1 that the integrals $d\lambda_2 d\lambda_4$ in (II.2) can be carried out to give, on top of the cut in S from Ψ_4 , a meromorphic function in s with double poles at $s = 2, 3, 4, \dots$.

It is now clear that double poles on top of cuts in s can only arise from regions where both $\lambda_2 \equiv |q|^{2u_2}$ and $\lambda_4 \equiv |q|^{2u_4}$ can approach 1 and where the exponent in $|q|^{-(Su_1 u_3 + T_0 u_2 u_4 + 2k u_1 + 2l u_3)}$ can be negative. This implies in particular that (II.1) is actually the *only* region which gives rise to such singularities. For example, in (II.1), $|q|^{2u_2}$ stays away from the radius of convergence of $F(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_2})$. Thus $F(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_2})$ is entire in s , and this region can contribute at most *simple* poles on top of cuts. As for region (I), we can divide it into three subregions (I.1), (I.2), and (I.3), in complete analogy with the division (D.) for (II). In (I.3), the exponent in $|q|^{-(Su_1 u_3 + T_0 u_2 u_4 + 2k u_1 + 2l u_3)}$ is bounded, and hence there are no cuts. In both (I.1) and (I.2), the term $F(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_4})$ is entire in s , and hence these regions produce again at most simple poles on top of cuts. We provide now some details of the argument.

Consider first (II.1). We can expand $F(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_2})$ in a uniformly convergent series, integrate first with respect to $|q|$, and then with respect to u_2 and λ_4 . In terms of the variables x and κ defined by

$$\begin{aligned} x &= 2(1-u_1)\tau_2 \\ |q|^{2u_2} &= 2^{-\kappa} [2|q|^{2(1-u_1)}]^{1-\kappa} \end{aligned}$$

we obtain

$$\begin{aligned} \text{(II.1)} &= \sum_{k_2=0}^{\infty} C_s(k_2) 2^{-k_2} \int_0^1 du_1 (1-u_1)^3 \int_0^1 d\kappa \, 2^{(\frac{1}{2}-\kappa)(-Su_1+2l)} \\ &\quad \times \int_{1/2}^1 d\lambda_4 \, \lambda_4^{-1+\frac{1}{2}Su_1-l-\frac{1}{2}T_0(1-u_1)(1-\kappa)} F(\frac{s}{2}, \frac{s}{2}; 1; \lambda_4) \\ &\quad \times \int_{\ln 2}^{\infty} \frac{dx}{x^4} (\ln 2 - x) \exp(-x((-Su_1 + 2l)\kappa + 2(1-\kappa)k_2) + 2k \frac{u_1}{1-u_1}) \\ &\quad \times 2^{2(1-\kappa)k_2} \exp(-\frac{1}{8x} T_0(1-u_1)(1-2\kappa)\ln(\lambda_4)\ln 2) \end{aligned} \tag{E.4}$$

Again we may rewrite the dx integral in terms of its analytic continuations Ψ_3 and Ψ_4 (c.f. Appendix F), which are holomorphic for $s \in \mathbf{C} \setminus \mathbf{R}_+$ and smooth in all the other parameters u_1, κ , and λ_4 . Furthermore, in the range of integration $x \geq \ln 2$, the factor

$$e^{-(1-\kappa)k_2(x-\ln 2)}$$

guarantees uniform bounds in k_2 . Thus the series in k_2 can be summed, and only simple poles due to the λ_4 integration can arise.

We turn next to (I.2). Here $F(\frac{s}{2}, \frac{s}{2}; 1; |q|^{2u_4})$ is entire in s and can be expanded in a uniformly convergent series in $|q|^{2u_4}$. Introducing the variables

$$|q|^{2u_4} = 2^{-\kappa} [2|q|^{2(1-u_1-u_2)}]^{1-\kappa} \quad (E.5)$$

we find

$$\begin{aligned} (I.2) &= \sum_{k_4=0}^{\infty} C_s(k_4) 2^{-k_4} \int_0^{\kappa} \int_{1/2}^1 d\lambda_2 \lambda_2^{-1-T_0(1-u_1)(1-\kappa)} F\left(\frac{s}{2}, \frac{s}{2}; 1; \lambda_2\right) \\ &\times \int_0^1 du_1 \theta(\ln 2 - 2\pi(1-u_1)) \lambda_2^{\frac{1}{2}((Su_1-2l)\kappa - T_0(1-u_1)(1-\kappa))} 2^{-\frac{\kappa}{2}(Su_1-2l)} \\ &\times \int_1^{\infty} \frac{dx}{x^4} (2(1-u_1)x - \ln \frac{\lambda_2}{2}) [2e^{-4\pi x(1-u_1)} \lambda_2^{-1}]^{k_4(1-\kappa)} \\ &\times \exp(-2\pi x(-Su_1 + 2l)(1-u_1)\kappa + \frac{1}{4\pi x} T_0((1-\kappa)(\ln \lambda_2)^2 + \kappa \ln \lambda_2 \ln 2)) \end{aligned}$$

As in the previous case, the analytic continuation in S can be obtained now via the holomorphic functions Ψ_3 and Ψ_4 of Appendix F. The series in k_4 is easily verified to be uniformly convergent, and there are at most simple poles on top of cuts, arising from the integration in λ_2 .

Finally we discuss (I.1). The bounds defining the region of integration

$$|q|^{2(1-u_1)} \leq \frac{1}{4}, \quad |q| \leq e^{-2\pi}$$

can be expressed using Heaviside functions as

$$\theta(\ln 2 - 2\pi(1-u_1))\theta(1 - 4|q|^{2(1-u_1)}) + \theta(2\pi(1-u_1) - \ln 2)\theta(1 - e^{2\pi}|q|)$$

On the support of the first term we keep the same variables $u_1, |q|$, while on the support of the second we change variables to $u_1, p \equiv |q|^{2(1-u_1)}$. The net effect is to split the $du_1 d\tau_2$ integral as

$$\int du_1 \int \frac{d\tau_2}{2\pi\tau_2} = \int_0^{1-\frac{\ln 2}{2\pi}} du_1 \int_0^{e^{-2\pi}} \frac{d|q|}{|q|} + \int_{1-\frac{\ln 2}{2\pi}}^1 du_1 [2(1-u_1)]^{-1} \int_0^{1/4} \frac{dp}{p}$$

Both terms can be treated in the same way. The additional factor $(1-u_1)^{-1}$ in the second term is harmless, because of the presence of $(1-u_1)^3$ in the integrand. As an example, we discuss the first term.

We expand both $F(\frac{s}{2}, \frac{s}{2}; 1; q^{2u_2})$ and $F(\frac{s}{2}, \frac{s}{2}; 1; q^{2u_4})$ in uniformly convergent series. Convenient variables are now κ defined as before by (E.5), and μ defined by

$$|q|^{2u_2} \equiv 2^{-\mu} (2|q|^{2(1-u_1)})^{1-\mu} = (4|q|^{2(1-u_1)})^{1-\mu} 2^{-1} \quad (E.6)$$

in terms of which we have

$$\begin{aligned}
(I.1) = & \sum_{k_2, k_4=0}^{\infty} C_s(k_2) C_s(k_4) 2^{-(k_2+k_4)} \int_0^1 d\kappa \int_0^1 d\mu \mu \int_0^1 du_1 [2(1-u_1)]^3 \\
& \times \int_{\ln 2}^{\infty} \frac{dx}{x^4} (\ln(4e^{-2\lambda}))^2 2^{-\mu(Su_1-2l)\kappa} 2^{\frac{T_0}{2}([(1-2\mu)(1-\kappa)+\kappa](1-\mu)(1-u_1)-(1-u_1)(1-2\mu)(1-\kappa)\mu)} \\
& \times \exp(-x[(-Su_1+2l)\kappa - T_0(1-\mu)(1-u_1)(1-\kappa)]\mu + 2k) \\
& \times (4e^{-2x})^{(1-\mu)k_2} (4e^{-2x})^{\mu(1-\kappa)k_4}
\end{aligned} \tag{E.7}$$

As before, the $d\lambda$ integral can be analytically continued as a holomorphic function in $s \in \mathbf{C} \setminus \mathbf{R}_+$, with uniform bounds, and the series in k_2, k_4 converge. This implies that (I.1) contributes no poles on top of cuts.

In summary, the double poles on tops of cuts come solely from the term (II.2), which we now work out explicitly.

Isolating the coefficients of the double poles.

The integral representing the leading singularities in (II.2) is complicated because it mixes the λ_2, λ_4 variables. However, for the coefficients of the double poles, these variables disentangle. Indeed we may write

$$\begin{aligned}
\exp\left(\frac{1}{2x}(1-u_1)(\ln\lambda_2)(\ln\lambda_4)T_0\right) = & \sum_{p=0}^N \frac{1}{p!} \left[\frac{(1-u_1)(\ln\lambda_2)(\ln\lambda_4)T_0}{2x} \right]^p \\
& + \frac{1}{N!} \left[\frac{(1-u_1)(\ln\lambda_2)(\ln\lambda_4)T_0}{2x} \right]^{N+1} \\
& \times \int_0^1 dt (1-t)^N e^{t\frac{1}{2x}(1-u_1)(\ln\lambda_2)(\ln\lambda_4)T_0}
\end{aligned}$$

If we substitute this expansion in (II.2), we note that the contribution of the Taylor remainder on the right hand side has no poles in the half-space $\text{Re } S \leq N+2$. Indeed each insertion of a factor $\ln\lambda$ in the integral

$$\int_{1/2}^1 d\lambda \lambda^{-1-\alpha} (\ln\lambda)^p F_s(\lambda)$$

pushes out the location of the first pole in s by 1, since $\ln\lambda$ vanishes of first order at $\lambda = 1$. This means that for S in any fixed half-space, the leading singularities of $A_{kl}(s; S, T_0)$ are of the form

$$\begin{aligned}
A(s; S, T_0) = & \frac{\pi}{2} \sum_{p=0}^N \frac{T_0^p}{p!} \int_0^1 du (1-u)^{3+p} \left(\int_{1/2}^1 d\lambda \lambda^{-1+\frac{1}{2}Su-k} (\ln\lambda)^p F_s(\lambda) \right)^2 \\
& \int_{\ln\sqrt{2}}^{\infty} \frac{dx}{2^p x^{p+4}} \exp\left(-x\left(-Su+2l+2k\frac{u}{1-u}\right)\right)
\end{aligned}$$

Since we are presently concerned only with the poles on top of cuts (equivalently the imaginary part of the coefficient or the decay rate), we can replace the dx integral by the singular part of the function $\Psi(s, 0)$ of Appendix A, i.e., by $c_n s^{n-1} \ln(-s)$. This means that the leading coefficients of the poles on top of cuts can be written entirely in terms of the coefficients d_{pm} defined by

$$\int_{1/2}^1 d\lambda \lambda^{-1-\alpha} (\ln \lambda)^p F_s(\lambda) = \sum_{m=2+p}^N \frac{d_{pm}}{s-m} + \dots$$

Appendix F: LOGARITHMIC CUTS

In this appendix we treat analytic continuations producing logarithmic cuts. We also formulate some bounds which are needed for convergence issues. The integrals which concern us here are of the form

$$L_n(s, t) = \int_{\alpha}^{\infty} \frac{dx}{x^n} e^{-2\pi x s + \frac{1}{8\pi x} t}$$

Lemma F.1. *a) The function $L_n(s, t)$ can be analytically continued as a holomorphic function of both s and t for*

$$s \in \mathbf{C} \setminus \mathbf{R}_+, \quad t \in \mathbf{C}$$

b) There exist constants C and, for each fixed compact set K , constants C_K so that

$$|L_n(s, t)| \leq C \left(\frac{|s|}{|\operatorname{Im} s|} \right)^{n-1} e^{|st|/2\pi\alpha |\operatorname{Im} s|}, \quad n > 1$$

$$|L_n(s, t)| \leq C_K \left(1 + \frac{|\operatorname{Re} s|}{|\operatorname{Im} s|} + \ln |s| \right), \quad s, t \in K$$

c) Furthermore

$$\begin{aligned} |L_n(s + \mu, t)| &\leq C e^{-2\pi\mu\alpha}, \quad \mu \geq 0, n > 1 \\ &\leq C e^{-2\pi\mu\alpha} \frac{1}{\mu} \ln \left(1 + \frac{\mu}{\alpha} \right), \quad \mu \geq 0, n = 1 \end{aligned}$$

Proof. To obtain the analytic continuation in (a), we note that for s a positive number, the integral can be rewritten as

$$e^{-2\pi\alpha s} s^{n-1} \int_0^{\infty} \frac{d\lambda}{(\lambda + \alpha s)^n} e^{-2\pi\lambda + \frac{st}{8\pi(\lambda + \alpha s)}} \quad (F.1)$$

It is evident that for s, t in the region described in (a), this expression converges, giving the desired analytic continuation. The remaining statements (b) and (c) are also easy to verify starting from (F.1).

For $t = 0$ we can write down explicit formulas for L_n in terms of logarithms. Thus

$$\int_1^\infty \frac{dx}{x} e^{-xs} = -\ln s + \int_0^1 \frac{dx}{x} (1 - e^{-xs}) + \int_0^1 \frac{dx}{x} (e^{-x} - 1) + \int_1^\infty \frac{dx}{x} e^{-x}$$

as can be seen by differentiating in s and integrating back. Similarly

$$\int_1^\infty \frac{dx}{x^2} e^{-xs} = -s \ln s + (1 + c_2)s - 1 + \int_0^1 \frac{dx}{x^2} (e^{-xs} - 1 + xs) \quad (F.2)$$

For later reference, we note that

$$\int_0^\infty dx \left(\frac{x^2}{(x+s)^2} - 1 + \frac{2s}{x+1} \right) = s + 2s \ln s$$

so that (A.2) can be rewritten as

$$\int_1^\infty \frac{dx}{x^2} e^{-xs} = - \int_0^1 \frac{dx}{x^2} (e^{-xs} - 1 + xs) + \frac{1}{2} \int_0^\infty dx \left(\frac{x^2}{(x+s)^2} - 1 + \frac{2s}{x+1} \right) - (c_2 + \frac{3}{2})s + 1 \quad (F.3)$$

More generally, we have

$$\int_1^\infty \frac{dx}{x^n} e^{-xs} = \frac{(-1)^{n-1} s^{n-1} \ln s}{\Gamma(n)} + \text{entire function}$$

The function $L_2(s, 0)$ is the one appearing in (7.15).

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FIGURE CAPTIONS

- Fig. 1 One loop superstring amplitude as an integral of vertex operators over the torus.
- Fig. 2 Singularities in the one loop integral representation
 - (a) when two vertex operators collide and $z_i \sim z_j$
 - (b) when $\tau_2 \mapsto \infty$ and the torus degenerates to a thin wire.
- Fig. 3 The off-shell two point function in quantum field theory.
- Fig. 4 One loop superstring singularities expected in the analytically continued amplitudes.
 - (a) branch cuts
 - (b) single poles with branch cuts on top of the pole
 - (c) double poles with real and imaginary parts to the two point function.
- Fig. 5 The box diagram in a ϕ^3 like theory with arbitrary mass assignments on all propagators.
- Fig. 6 Constant f contours and location of extrema.
- Fig. 7 Support of the spectral density function $\rho_{\{m_i^2\}}$.
- Fig. 8 Duality and its connection with analytic continuation to tree level.
- Fig. 9 Appearance of poles due to analytic continuation of an infinite sum of box diagrams.

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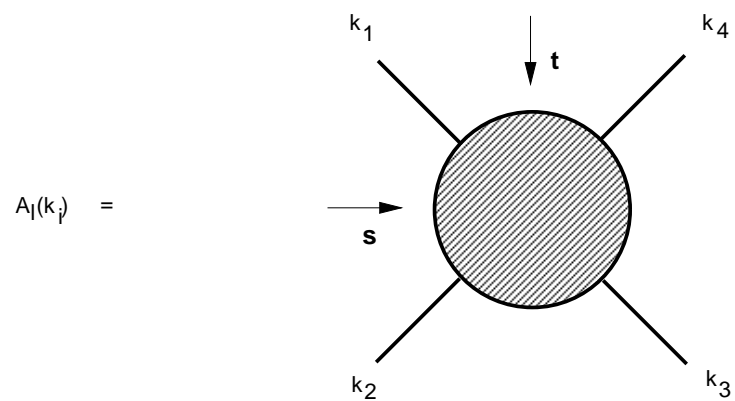


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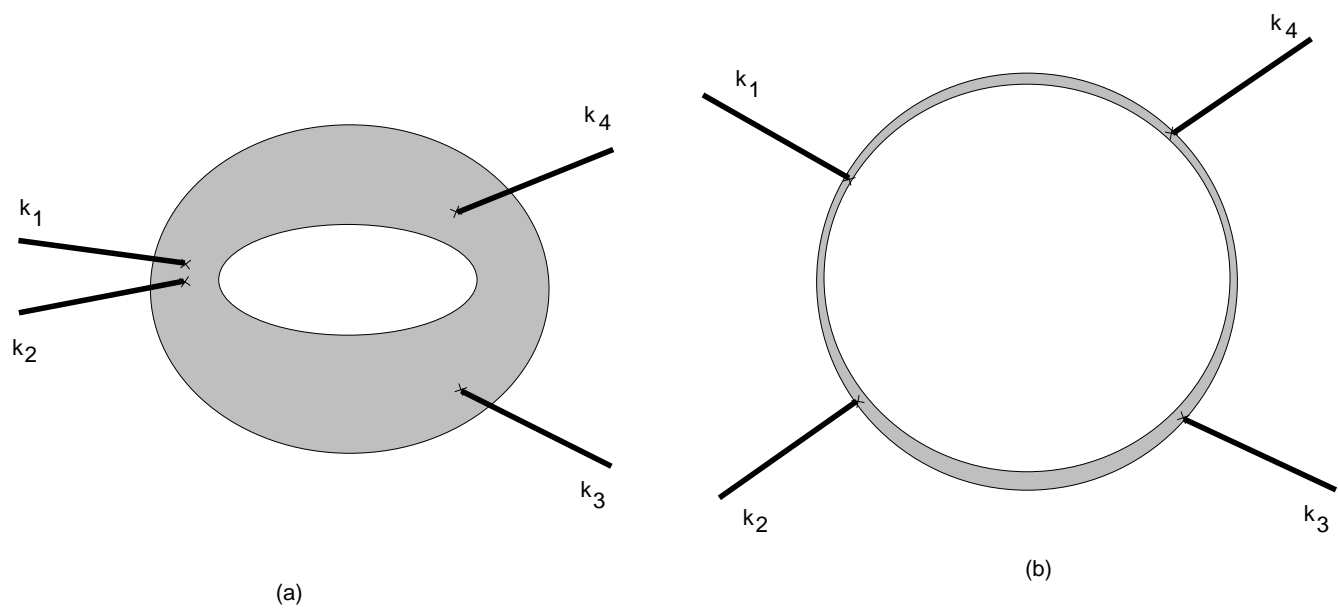


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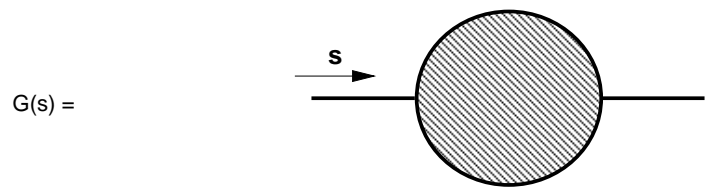


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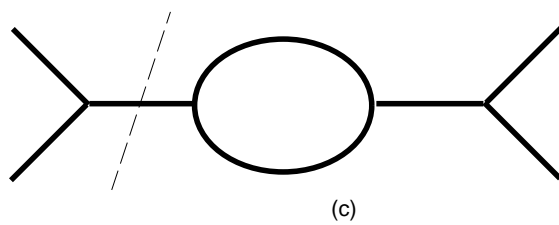
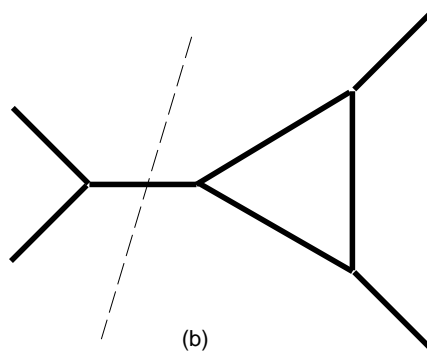
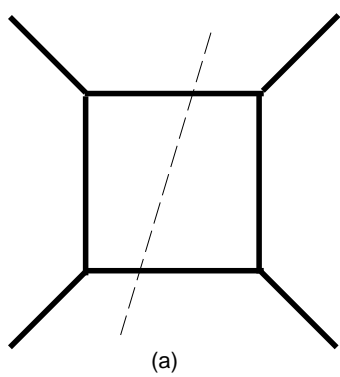


Fig. 4

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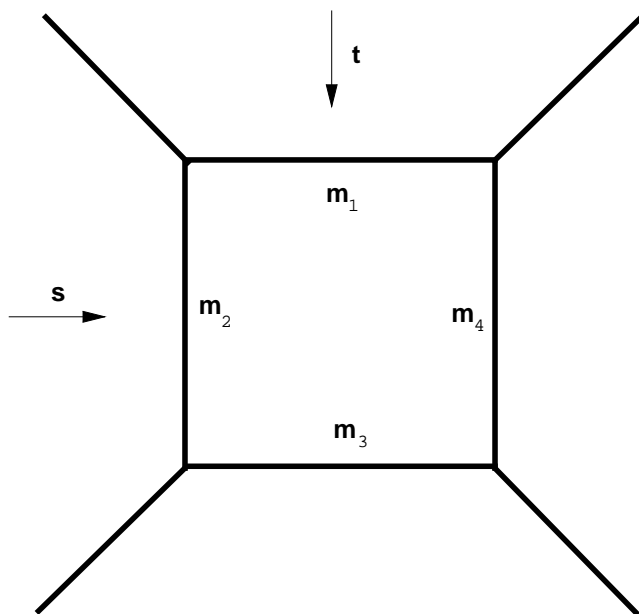


Fig. 5

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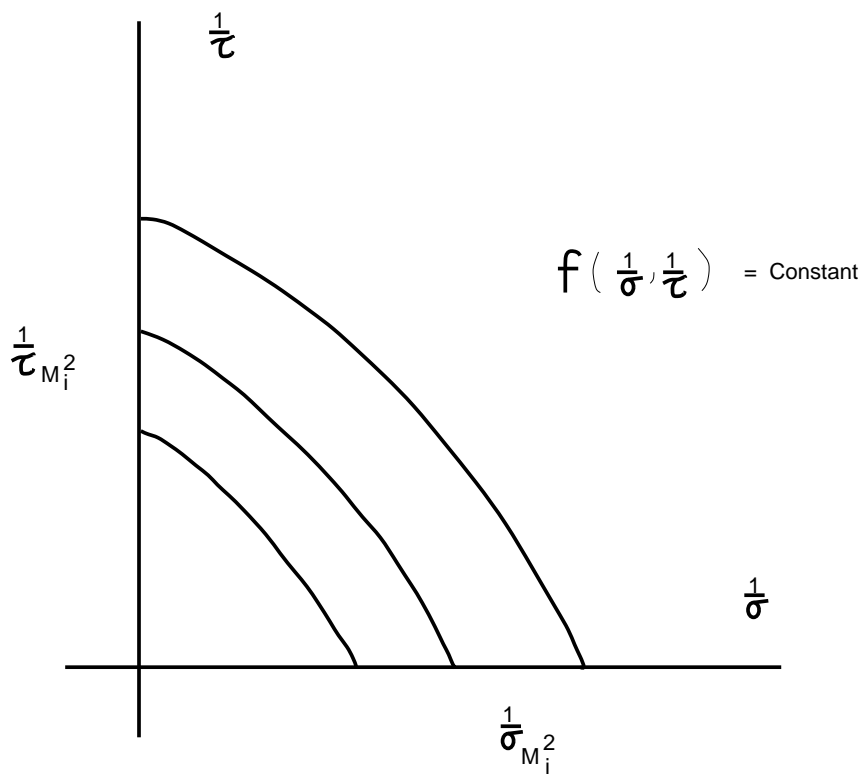


Fig. 6

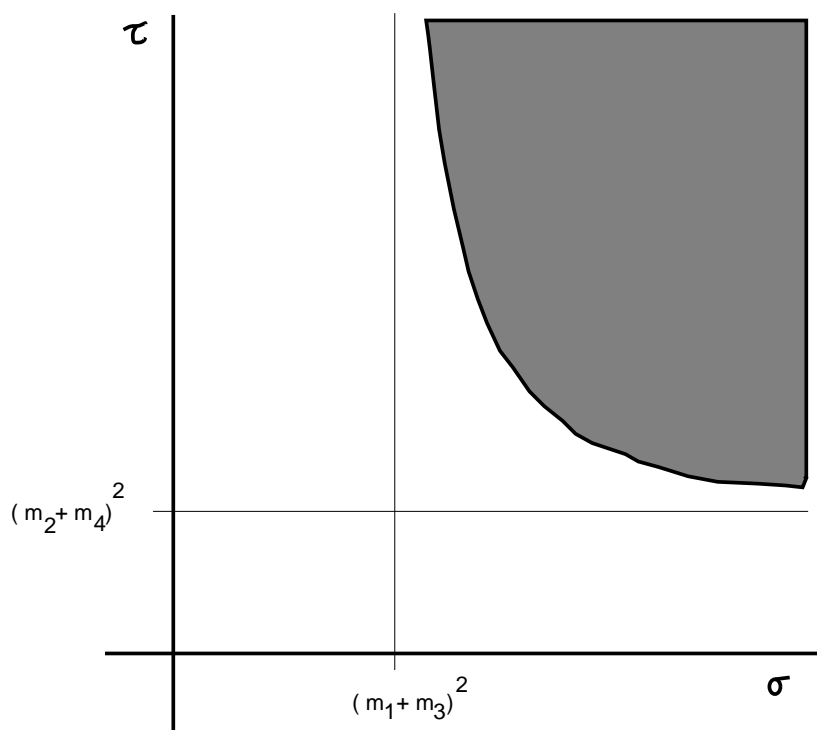


Fig. 7

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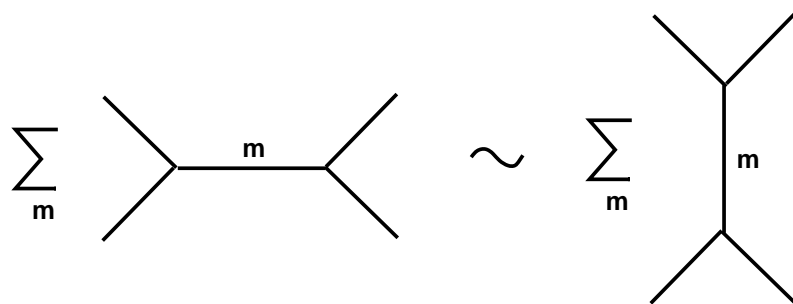


Fig. 8

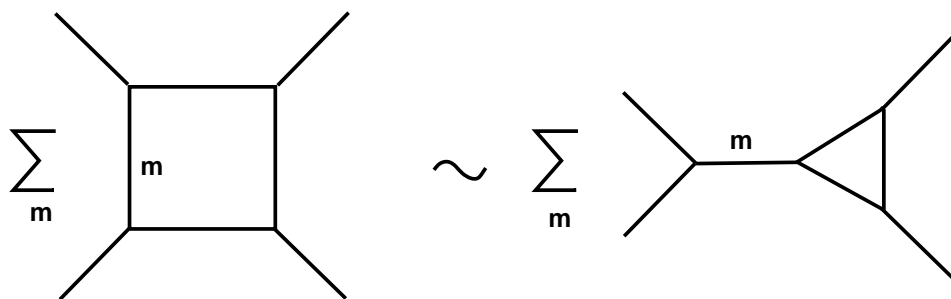


Fig. 9